

ON THE OHNO-NAKAGAWA THEOREM

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ABSTRACT. In this paper we give a new proof of the Ohno-Nakagawa Theorem using the techniques of L -series. By applying Eisenstein's parametrization of binary cubic forms on the one hand, and a class field theory interpretation of Datskovsky & Wright's Theorem on the other, we reduce the Ohno-Nakagawa Theorem to an identity involving the L -series and the truncated L -series of quadratic orders. We prove this identity by establishing a general relation among these two types L -series.

1. INTRODUCTION

Let L denote the lattice of integral binary cubic forms:

$$L = \{ x(u, v) = x_0 u^3 + x_1 u^2 v + x_2 u v^2 + x_3 v^3 \mid x_i \in \mathbb{Z}, 0 \leq i \leq 3 \}.$$

Put $\Gamma = SL(2, \mathbb{Z})$. The group Γ acts on L by the formula

$$(\gamma x)(u, v) = x((u, v)\gamma), \quad \forall \gamma \in \Gamma, x \in L.$$

Let $\text{disc}(x)$ denote the discriminant of $x(u, v)$, i.e.,

$$\text{disc}(x) = x_1^2 x_2^2 + 18 x_0 x_1 x_2 x_3 - 4 x_0 x_2^3 - 4 x_1^3 x_3 - 27 x_0^2 x_3^2.$$

It is invariant under the action of Γ . For $x \in L$, let Γ_x denote the isotropic subgroup of x in Γ . It is known that $|\Gamma_x|$ is 1 if $\text{disc}(x) < 0$ and is 1 or 3 if $\text{disc}(x) > 0$. Let

$$L^\vee = \{ x(u, v) = x_0 u^3 + x_1 u^2 v + x_2 u v^2 + x_3 v^3 \in L \mid x_1, x_2 \in 3\mathbb{Z} \}$$

be the dual lattice of L with respect to the invariant alternating form:

$$\langle x, y \rangle = x_3 y_0 - \frac{1}{3} x_2 y_1 + \frac{1}{3} x_1 y_2 - x_0 y_3.$$

It is clear that L^\vee is an Γ -invariant lattice. Put

$$L_\pm = \{ x \in L \mid \pm \text{disc}(x) > 0 \} \quad \text{and} \quad L_\pm^\vee = \{ x \in L^\vee \mid \pm \text{disc}(x) > 0 \}.$$

In a seminal paper [17], Shintani introduced the following Dirichlet series now bearing his name:

$$\begin{aligned} \xi_1(s) &= \sum_{x \in \Gamma \backslash L_+} \frac{1}{|\Gamma_x|} |\text{disc}(x)|^{-s}, & \xi_2(s) &= \sum_{x \in \Gamma \backslash L_-} |\text{disc}(x)|^{-s}, \\ \xi_1^\vee(s) &= \sum_{x \in \Gamma \backslash L_+^\vee} \frac{1}{|\Gamma_x|} |\text{disc}(x)/27|^{-s}, & \xi_2^\vee(s) &= \sum_{x \in \Gamma \backslash L_-^\vee} |\text{disc}(x)/27|^{-s}. \end{aligned}$$

Date: November 30, 2015.

Key words and phrases. order, conductor, L -function, binary cubic form.

Here we scale Shintani's original series in the dual lattice case by a factor of 3^{3s} . Using the theory of prehomogeneous vector space, Shintani proved that all these series can be analytically continued to the whole complex plane with simple poles at 1 and $5/6$, and satisfy the matrix functional equation

$$\begin{pmatrix} \xi_1(1-s) \\ \xi_2(1-s) \end{pmatrix} = 2^{-1} 3^{3s-2} \pi^{-4s} \Gamma\left(s - \frac{1}{6}\right) \Gamma(s)^2 \Gamma\left(s + \frac{1}{6}\right) \\ \times \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3 \sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \xi_1^\vee(s) \\ \xi_2^\vee(s) \end{pmatrix}.$$

In 1997, Ohno made the surprising discovery that Shintani's four zeta functions are essentially two functions [13]. This was later proved by Nakagawa [11].

Theorem 1.1 (Nakagawa).

$$(1) \quad \xi_1^\vee(s) = \xi_2(s) ; \quad (2) \quad \xi_2^\vee(s) = 3 \xi_1(s).$$

Nakagawa's proof uses very sophisticated counting arguments. In this paper, we give a more streamlined proof by exploiting the rich algebraic structure lying beneath the four zeta functions. In fact, we show that identities (1) and (2) above can be further divided into infinitely many identities involving the L -series and the truncated L -series of quadratic orders. For other works related to these two mysterious identities, see [2, 3, 6, 14, 15, 18].

This paper is organized as follows. In section 2, we introduce the notations and definitions. In section 3, we give a self-contained introduction to Eisenstein's parametrization of binary cubic forms. Using this parametrization, we can express the $\xi_i^\vee(s)$'s directly in terms of L -series of quadratic orders. Note that in [11] Nakagawa has to resort to an analog of Datskovsky & Wright's formula to start the counting. In section 4, using the techniques of class field theory, we interpret Datskovsky & Wright's Theorem in terms of the truncated L -series of quadratic orders. This reduces the proof of Theorem 1.1 to an identity involving the L -series and the truncated L -series of quadratic orders. We prove this identity in section 5 by establishing a general relation between these two types L -series. In the Appendix, we derive a simple relation connecting the abelian L -series of a number field k and the L -series of certain orders of k^{n+1} . This is inspired by a problem arising in Section 4.

2. BASIC DEFINITIONS

Throughout this paper, we denote the cardinality of a finite set X by $|X|$. We let \mathbb{Z} denote the ring of integers and \mathbb{Z}^+ its subset of positive integers. Let (a, b) denote the greatest common divisor of integers a and b . Let $\text{sgn}(a)$ denote the sign of a real number a . If $\varphi : A \rightarrow B$ is a map and H is a subset of B , we write $\varphi^{-1}(H)$ for the inverse image of H . If G is an Abelian group, we let $G^\vee = \text{Hom}(G, \mathbb{C}^*)$ denote its dual group. We assume that all rings R have the identity 1 and all subrings of R share the same identity element of R . Furthermore, we let R^* denote the group formed by the invertible elements of R .

Orders of étale algebras. Let A be an étale algebra over \mathbb{Q} . This means that $A \simeq K_1 \times \cdots \times K_m$, where K_i ($1 \leq i \leq m$) are finite extensions of \mathbb{Q} . We let $\text{Tr}_{A/\mathbb{Q}}$ denote the trace map and $N_{A/\mathbb{Q}}$ the norm map from A to \mathbb{Q} . Let \mathcal{O}_A denote the unique maximal order of A , and Δ_A the discriminant of A . Thus

$\mathcal{O}_A \simeq \mathcal{O}_{K_1} \times \cdots \times \mathcal{O}_{K_m}$ and $\Delta_A = \Delta_{K_1} \cdots \Delta_{K_m}$, where \mathcal{O}_{K_i} is the maximal order of K_i , and Δ_{K_i} is the discriminant of K_i , $1 \leq i \leq m$.

If \mathfrak{a} and \mathfrak{b} are \mathbb{Z} -submodules of A , we let $\mathfrak{a}\mathfrak{b}$ denote the \mathbb{Z} -submodules of A generated by elements of the form $\alpha\beta$ with $\alpha \in \mathfrak{a}$ and $\beta \in \mathfrak{b}$. We call a finitely generated \mathbb{Z} -submodule \mathfrak{a} of A a full \mathbb{Z} -module if the \mathbb{Z} -rank of \mathfrak{a} is equal to the \mathbb{Q} -dimension of A . If \mathfrak{a} is a full \mathbb{Z} -module of A , we write

$$\mathfrak{a}^\vee = \{ \beta \in A \mid \text{Tr}_{A/\mathbb{Q}}(\alpha\beta) \in \mathbb{Z}, \forall \alpha \in \mathfrak{a} \}$$

for its dual full \mathbb{Z} -module.

We call a subring \mathcal{O} of \mathcal{O}_A an order of A if \mathcal{O} is a full \mathbb{Z} -module. Ideals of the ring \mathcal{O} are referred to integral ideals of \mathcal{O} . We call a full \mathbb{Z} -module \mathfrak{a} of A a (fractional) \mathcal{O} -ideal if $\mathcal{O}\mathfrak{a} = \mathfrak{a}$. We say an \mathcal{O} -ideal \mathfrak{a} is \mathcal{O} -invertible if there exists an \mathcal{O} -ideal \mathfrak{b} such that $\mathfrak{a}\mathfrak{b} = \mathcal{O}$. Observe that \mathcal{O}_A -ideals are always \mathcal{O}_A -invertible. The set of invertible \mathcal{O} -ideals forms a group under ideal multiplication. We denote this group by $I(\mathcal{O})$. Moreover, let $P(\mathcal{O})$ denote the subgroup of $I(\mathcal{O})$ consisting of principal \mathcal{O} -ideals $\alpha\mathcal{O}$ with $\alpha \in A^*$. The quotient group $I(\mathcal{O})/P(\mathcal{O})$ is called the ideal class group of \mathcal{O} and is denoted by $Cl(\mathcal{O})$. If \mathfrak{a} is an invertible \mathcal{O} -ideal, we let $[\mathfrak{a}]$ denote its ideal class in $Cl(\mathcal{O})$.

Let \mathfrak{c} be an integral ideal of an order \mathcal{O} . We say an integral ideal \mathfrak{a} of \mathcal{O} is prime to \mathfrak{c} if $\mathfrak{a} + \mathfrak{c} = \mathcal{O}$. We say that an invertible \mathcal{O} -ideal \mathfrak{a} is prime to \mathfrak{c} if \mathfrak{a} can be written as $\mathfrak{a}_1/\mathfrak{a}_2$, where \mathfrak{a}_1 and \mathfrak{a}_2 are integral invertible ideals of \mathcal{O} prime to \mathfrak{c} . All invertible \mathcal{O} -ideals prime to \mathfrak{c} form a group which is denoted by $I(\mathcal{O}, \mathfrak{c})$. The conductor of an order \mathcal{O} of A is defined by

$$\mathfrak{f} = \{ \gamma \in A \mid \gamma\mathcal{O}_A \subseteq \mathcal{O} \}.$$

It is the largest integral ideal of \mathcal{O}_A contained in \mathcal{O} . An integral \mathcal{O} -ideal \mathfrak{a} which is prime to the conductor \mathfrak{f} of \mathcal{O} is \mathcal{O} -invertible with inverse $\mathfrak{a}^{-1} = \mathcal{O} + \mathfrak{f}(\mathfrak{a}\mathcal{O}_A)^{-1}$.

Lemma 2.1. (*Dedekind*) *The map $\tilde{\mathfrak{a}} \mapsto \tilde{\mathfrak{a}} \cap \mathcal{O}$ is a bijection between the set of integral \mathcal{O}_A -ideals prime to the conductor \mathfrak{f} of \mathcal{O} and the set of integral \mathcal{O} -ideals prime to \mathfrak{f} . The inverse map is given by $\mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_A$.*

We can extend the map $\tilde{\mathfrak{a}} \mapsto \tilde{\mathfrak{a}} \cap \mathcal{O}$ to a homomorphism $\phi : I(\mathcal{O}_A, \mathfrak{f}) \rightarrow I(\mathcal{O}, \mathfrak{f})$ as follows. Take $\tilde{\mathfrak{a}} \in I(\mathcal{O}_A, \mathfrak{f})$ and write it as $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_1/\tilde{\mathfrak{a}}_2$, where $\tilde{\mathfrak{a}}_1$ and $\tilde{\mathfrak{a}}_2$ are integral ideals of \mathcal{O}_A prime to \mathfrak{f} . Put $\mathfrak{a}_i = \tilde{\mathfrak{a}}_i \cap \mathcal{O}$, $i = 1, 2$, and define $\phi(\tilde{\mathfrak{a}}) = \mathfrak{a}_1/\mathfrak{a}_2$. It is easy to see that ϕ is well defined.

Corollary 2.2. *The map $\phi : I(\mathcal{O}_A, \mathfrak{f}) \rightarrow I(\mathcal{O}, \mathfrak{f})$ is an isomorphism. Its inverse map is given by $\phi^{-1} : \mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_A$, $\forall \mathfrak{a} \in I(\mathcal{O}, \mathfrak{f})$.*

Put $P(\mathcal{O}, \mathfrak{f}) = I(\mathcal{O}, \mathfrak{f}) \cap P(\mathcal{O})$. Then we have

Lemma 2.3. $Cl(\mathcal{O}) \simeq I(\mathcal{O}, \mathfrak{f})/P(\mathcal{O}, \mathfrak{f})$.

Let \mathcal{O} and \mathcal{O}' be orders of A such that $\mathcal{O} \subseteq \mathcal{O}'$. We say that an \mathcal{O} -ideal \mathfrak{a} lies under an \mathcal{O}' -ideal \mathfrak{a}' if $\mathfrak{a}\mathcal{O}' = \mathfrak{a}'$. We let $U(\mathcal{O}', \mathcal{O})$ denote the group formed by those invertible \mathcal{O} -ideals lying under \mathcal{O}' . Let $\varphi_{\mathcal{O}', \mathcal{O}} : Cl(\mathcal{O}) \rightarrow Cl(\mathcal{O}')$ denote the homomorphism sending the \mathcal{O} -ideal class $[\mathfrak{a}]$ to the \mathcal{O}' -ideal class $[\mathfrak{a}\mathcal{O}']$, for every $\mathfrak{a} \in I(\mathcal{O})$. The homomorphism $\varphi_{\mathcal{O}', \mathcal{O}}$ is surjective and induces the exact sequence

$$1 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{O}'^* \longrightarrow U(\mathcal{O}', \mathcal{O}) \longrightarrow Cl(\mathcal{O}) \longrightarrow Cl(\mathcal{O}') \longrightarrow 1.$$

From this we deduce that

$$(2.1) \quad |U(\mathcal{O}', \mathcal{O})| = [\mathcal{O}'^* : \mathcal{O}^*] |Cl(\mathcal{O})| / |Cl(\mathcal{O}')|.$$

If \mathfrak{a} is an integral invertible ideal of \mathcal{O} , then $[\mathcal{O} : \mathfrak{a}] = [\mathcal{O}_A : \mathfrak{a}\mathcal{O}_A]$. We call this index the norm of \mathfrak{a} , and denote it by $N\mathfrak{a}$. It is easy to extend the definition of norm to all invertible ideals of \mathcal{O} .

Let \mathcal{O} be an order of A of conductor \mathfrak{f} and let \mathfrak{A} be an ideal class of $Cl(\mathcal{O})$. We define the partial zeta function

$$(2.2) \quad \zeta(s, \mathfrak{A}, \mathcal{O}) = \sum_{\mathfrak{a} \in \mathfrak{A}} N\mathfrak{a}^{-s},$$

where \mathfrak{a} goes through all integral invertible \mathcal{O} -ideals in the ideal class \mathfrak{A} . We also define the truncated partial zeta function as

$$(2.3) \quad \zeta^*(s, \mathfrak{A}, \mathcal{O}) = \sum_{\mathfrak{b}} N\mathfrak{b}^{-s},$$

where \mathfrak{b} ranges over all integral \mathcal{O} -ideals in \mathfrak{A} which are prime to the conductor \mathfrak{f} . To each character χ of the class group $Cl(\mathcal{O})$, we associate the L -series

$$L(s, \chi, \mathcal{O}) = \sum_{\mathfrak{A} \in Cl(\mathcal{O})} \chi(\mathfrak{A}) \zeta(s, \mathfrak{A}, \mathcal{O}) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) N\mathfrak{a}^{-s}$$

and the truncated L -series

$$L^*(s, \chi, \mathcal{O}) = \sum_{\mathfrak{A} \in Cl(\mathcal{O})} \chi(\mathfrak{A}) \zeta^*(s, \mathfrak{A}, \mathcal{O}).$$

Since an integral prime-to- \mathfrak{f} \mathcal{O} -ideal \mathfrak{b} factors uniquely into prime ideals of \mathcal{O} in exactly the same way as $\mathfrak{b}\mathcal{O}_A$ does in \mathcal{O}_A , we have the Euler product expansion

$$L^*(s, \chi, \mathcal{O}) = \sum_{(\mathfrak{b}, \mathfrak{f})=1} \chi(\mathfrak{b}) N\mathfrak{b}^{-s} = \prod_{\mathfrak{p} \nmid \mathfrak{f}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1},$$

where the product runs over prime ideals \mathfrak{p} of \mathcal{O} not dividing the conductor \mathfrak{f} .

Characters and conductors of class groups. Let k be a number field and let \mathfrak{m} be a modulus in k , that is, \mathfrak{m} is a formal product $\mathfrak{m}_0 \mathfrak{m}_\infty$ where \mathfrak{m}_0 is an integral ideal of \mathcal{O}_k and \mathfrak{m}_∞ is a set of real embeddings of k . Let $\alpha \in k$. We say that $\alpha \equiv 1 \pmod{* \mathfrak{m}}$ if $v_{\mathfrak{p}}(\alpha - 1) \geq v_{\mathfrak{p}}(\mathfrak{m}_0)$ for all primes \mathfrak{p} of \mathcal{O}_k dividing \mathfrak{m}_0 , and if $\sigma(\alpha) > 0$ for all embeddings $\sigma \in \mathfrak{m}_\infty$. We let $I_k(\mathfrak{m}) = I(\mathcal{O}_k, \mathfrak{m}_0)$ denote the group formed by those \mathcal{O}_k -ideals prime to \mathfrak{m}_0 , and let $P_{k,1}(\mathfrak{m})$ denote its subgroup consisting of principal ideals $\alpha\mathcal{O}_k$ with $\alpha \equiv 1 \pmod{* \mathfrak{m}}$. The quotient group $I_k(\mathfrak{m})/P_{k,1}(\mathfrak{m})$ is called the ray class group of k modulo \mathfrak{m} , and is denoted by $Cl_k(\mathfrak{m})$.

For each pair of moduli $\mathfrak{m} = \mathfrak{m}_0 \mathfrak{m}_\infty$ and $\mathfrak{m}' = \mathfrak{m}'_0 \mathfrak{m}'_\infty$ with $\mathfrak{m}' | \mathfrak{m}$, (i.e., $\mathfrak{m}'_0 | \mathfrak{m}_0$ and $\mathfrak{m}'_\infty \subseteq \mathfrak{m}_\infty$), we let $\psi_{k, \mathfrak{m}', \mathfrak{m}}$ denote the surjective homomorphism

$$Cl_k(\mathfrak{m}) = I_k(\mathfrak{m})/P_{k,1}(\mathfrak{m}) \rightarrow Cl_k(\mathfrak{m}') = I_k(\mathfrak{m}')/P_{k,1}(\mathfrak{m}')$$

induced by the inclusion map $I_k(\mathfrak{m}) \subseteq I_k(\mathfrak{m}')$. If χ' is a character of $Cl_k(\mathfrak{m}')$, then $\chi = \chi' \circ \psi_{k, \mathfrak{m}', \mathfrak{m}}$ is a character of $Cl_k(\mathfrak{m})$. We say in this case that χ is induced by χ' , or χ is defined at $I_k(\mathfrak{m}')$. If a character χ of $Cl_k(\mathfrak{m})$ cannot be induced by any character χ' of $Cl_k(\mathfrak{m}')$ for any proper divisor \mathfrak{m}' of \mathfrak{m} , then we say that χ is a primitive character of $Cl_k(\mathfrak{m})$. It is a standard fact in class field theory that every character χ of $Cl_k(\mathfrak{m})$ is induced by a unique primitive character of $Cl_k(\mathfrak{f})$ for some modulus \mathfrak{f} . The modulus \mathfrak{f} is uniquely determined by χ and is called the conductor of χ .

Similarly, let H' be a subgroup of $Cl_k(\mathfrak{m}')$ and let \mathfrak{m} be a modulus in k divisible by \mathfrak{m}' , then $H = \psi_{k,\mathfrak{m}',\mathfrak{m}}^{-1}(H')$ is a subgroup of $Cl_k(\mathfrak{m})$. We say that the subgroup H of $Cl_k(\mathfrak{m})$ is induced by the subgroup H' of $Cl_k(\mathfrak{m}')$. For each subgroup H of $Cl_k(\mathfrak{m})$, there is a unique modulus \mathfrak{f} and a unique subgroup H_1 of $Cl_k(\mathfrak{f})$ such that H is induced by H_1 and H_1 cannot be further induced by any other subgroup of $Cl_k(\mathfrak{m}')$ for any modulus \mathfrak{m}' . The modulus \mathfrak{f} is called the conductor of H .

More generally, let (I, \prec) be a partially ordered set. We assume for simplicity that for each $j \in I$, the number of indexes $i \in I$ with $i \prec j$ is finite. Let $(A_i)_{i \in I}$ be a family of finite abelian groups and suppose we have a family of surjective homomorphisms $\varphi_{ij} : A_j \rightarrow A_i$ for all $i \prec j$ with the following properties :

- 1) φ_{ii} is the identity map on A_i ;
- 2) $\varphi_{im} = \varphi_{ij} \circ \varphi_{jm}$ for all $i \prec j \prec m$;
- 3) If $i \prec m$ and $j \prec m$, then there exists a $l \in I$ with $l \prec i$ and $l \prec j$ such that $\text{Ker } \varphi_{im} \text{ Ker } \varphi_{jm} = \text{Ker } \varphi_{lm}$.

Then the pair $((A_i)_{i \in I}, (\varphi_{ij})_{i \prec j \in I})$ is called a compatible inverse system of abelian groups over I , and the homomorphisms φ_{ij} are called the transition morphism of the system. We say that a subgroup H of A_j is induced by a subgroup H' of A_i if $i \prec j$ and $\varphi_{ij}^{-1}(H') = H$. Clearly a subgroup H of A_j is induced from a subgroup of A_i if and only if $i \prec j$ and $\text{Ker } \varphi_{ij} \subseteq H$. Thus in a compatible inverse system every subgroup H of A_j is induced by a unique subgroup H_1 of A_f for some $f \prec i$ such that H_1 cannot be further induced by any other subgroups except itself. Such index $f \in I$ is unique and is called the conductor of the subgroup H . One defines the conductors of characters of A_j in a similar way. In fact, the conductor of a character χ of A_j is simply the conductor of the subgroup $\text{Ker } \chi$ of A_j .

As an example let k be a quadratic étale algebra over \mathbb{Q} , i.e., k is either a quadratic field or $\mathbb{Q} \oplus \mathbb{Q}$. Let \mathcal{O}_k denote the maximal order of k and write $\mathbf{1}$ for the identity element of k . For $m \in \mathbb{Z}^+$, we let $\mathcal{O}_{k,m} = \mathbb{Z}\mathbf{1} + m\mathcal{O}_k$ denote the order of k of conductor m . It is easy to see that for $c, d \in \mathbb{Z}^+$, we have $\mathcal{O}_{k,c}\mathcal{O}_{k,d} = \mathcal{O}_{k,l}$, where $l = (c, d)$. Write $Cl_{k,m} = Cl(\mathcal{O}_{k,m})$ for the class group of $\mathcal{O}_{k,m}$. Note that in the case $k = \mathbb{Q}^2$, each class of $Cl_{\mathbb{Q}^2,m}$ contains an invertible $\mathcal{O}_{\mathbb{Q}^2,m}$ -ideal of the form $\mathfrak{a}_t = \mathbb{Z}(1, t) + \mathbb{Z}(0, m)$, where t is an integer prime to m , and two such ideals \mathfrak{a}_t and \mathfrak{a}_s represent the same class precisely when $s \equiv \pm t \pmod{m}$. Since we have $\mathfrak{a}_1 = \mathcal{O}_{\mathbb{Q}^2,m}$ and $\mathfrak{a}_s \mathfrak{a}_t = \mathfrak{a}_{st}$, the map sending $[\mathfrak{a}_t]$ to $\pm t \pmod{m}$ is an isomorphism $Cl_{\mathbb{Q}^2,m} \cong (\mathbb{Z}/m\mathbb{Z})^* / \{\pm 1\}$ (see also Lemma A.2).

For $m, m' \in \mathbb{Z}^+$ with $m' | m$, let $\varphi_{k,m',m} : Cl_{k,m} \rightarrow Cl_{k,m'}$ denote the surjective homomorphism sending the $\mathcal{O}_{k,m}$ -ideal class $[\mathfrak{a}]$ to the $\mathcal{O}_{k,m'}$ -ideal class $[\mathfrak{a}\mathcal{O}_{k,m'}]$, for every $\mathfrak{a} \in I(\mathcal{O}_{k,m})$. The following Lemma shows that for characters of the class groups of quadratic orders, the concept of conductor is well defined.

Lemma 2.4. *Let k be a quadratic étale algebra. Then $((Cl_{k,m})_{m \in \mathbb{Z}^+}, (\varphi_{k,m',m}))$ forms a compatible inverse system over \mathbb{Z}^+ partially ordered by the division relation.*

Proof. It suffices to check condition (3) of the compatible inverse system. We claim that for c, d and $m \in \mathbb{Z}^+$ with $c | m$ and $d | m$,

$$\text{Ker } \varphi_{k,c,m} \text{ Ker } \varphi_{k,d,m} = \text{Ker } \varphi_{k,l,m},$$

where $l = (c, d)$. Take any $\mathcal{O}_{k,m}$ -ideal class in $\text{Ker } \varphi_{k,l,m}$. Without loss of generality, we assume that it is represented by an invertible $\mathcal{O}_{k,m}$ -ideal \mathfrak{a} with $\mathfrak{a}\mathcal{O}_{k,l} = \mathcal{O}_{k,l}$. Choose $u, v \in \mathbb{Z}^+$ with $c | ul$ and $d | vl$ such that $(u, v) = 1$ and $uvl = m$. Put

$$\mathfrak{a}_1 = u\mathfrak{a}\mathcal{O}_{k,vl} + v\mathcal{O}_{k,ul} \quad \text{and} \quad \mathfrak{a}_2 = v\mathfrak{a}\mathcal{O}_{k,ul} + u\mathcal{O}_{k,vl}.$$

It is clear that \mathfrak{a}_1 and \mathfrak{a}_2 are $\mathcal{O}_{k,m}$ -ideals. They satisfy

$$\begin{aligned}\mathfrak{a}_1 \mathfrak{a}_2 &= uv \mathfrak{a}^2 \mathcal{O}_{k,l} + u^2 \mathfrak{a} \mathcal{O}_{k,ul} + v^2 \mathfrak{a} \mathcal{O}_{k,ul} + uv \mathcal{O}_{k,l} \\ &= \mathfrak{a} (uv \mathcal{O}_{k,l} + u^2 \mathbb{Z} \mathbf{1} + u^2 vl \mathcal{O}_k + v^2 \mathbb{Z} \mathbf{1} + v^2 ul \mathcal{O}_k) \\ &= \mathfrak{a} (\mathbb{Z} \mathbf{1} + m \mathcal{O}_k) = \mathfrak{a}.\end{aligned}$$

This implies that \mathfrak{a}_1 and \mathfrak{a}_2 are invertible $\mathcal{O}_{k,m}$ -ideals. Moreover, we have

$$\begin{aligned}\mathfrak{a}_1 \mathcal{O}_{k,c} &= u \mathfrak{a} \mathcal{O}_{k,l} + v \mathcal{O}_{k,c} \\ &= u \mathbb{Z} \mathbf{1} + ul \mathcal{O}_k + v \mathbb{Z} \mathbf{1} + vc \mathcal{O}_k \\ &= \mathcal{O}_{k,c}\end{aligned}$$

and similarly $\mathfrak{a}_2 \mathcal{O}_{k,d} = \mathcal{O}_{k,d}$. Hence $[\mathfrak{a}] = [\mathfrak{a}_1][\mathfrak{a}_2] \in \text{Ker } \varphi_{k,c,m} \text{ Ker } \varphi_{k,d,m}$. The other side of the inclusion is trivial. \square

3. INTERPRETING $\xi_i^\vee(s)$ IN TERMS OF L -SERIES OF QUADRATIC ORDERS

Throughout the section, we let k denote a quadratic étale algebra, i.e., k is either a quadratic field or $\mathbb{Q} \oplus \mathbb{Q}$. We define the quadratic order $\mathcal{O}_{k,m}$ and its class group $Cl_{k,m}$ as before. For $\alpha \in k$, we write α^τ for the image of α under the unique non-trivial automorphism τ of k . When k is a quadratic field, we let $\sqrt{\Delta_k}$ denote the positive square root of Δ_k if $\Delta_k > 0$ and the positive imaginary square root if $\Delta_k < 0$. In the case $k = \mathbb{Q} \oplus \mathbb{Q}$, we use $\sqrt{\Delta_k}$ to denote the element $(-1, 1) \in k$, and identify \mathbb{Q} as a subalgebra of k via the diagonal map $a \mapsto (a, a)$, $\forall a \in \mathbb{Q}$.

Integral binary cubic forms with middle coefficients divisible by 3. In this section let n be a non-zero integer. We want to count the number of Γ -orbits in

$$L^\vee(n) = \{x \in L^\vee \mid \text{disc}(x) = -27n\}.$$

As was shown by Eisenstein [7] in the 19th century, binary cubic forms of this type are parameterized by ideal classes of their quadratic resolvent rings. Eisenstein proved this for certain maximal orders. Nakagawa made this correspondence explicit in [11, Section 2] for orders of imaginary quadratic fields. Bhargava constructed in his doctoral thesis [1, Thm 13] a bijective correspondence for general quadratic rings. He associated to each Γ -orbit in L^\vee an equivalent class of the triples (S, I, δ) , where S is an oriented quadratic ring, I is an ideal of S , and δ is an element of $S \otimes \mathbb{Q}$ satisfying certain compatibility conditions. Since the ideal I is in most cases not S -invertible, one has to re-group these triples to get to conditions on ideal class groups. In the following we give a self-contained account of the parametrization, with proof based essentially on one equation—the refined syzygy identity (3.4). We also state the bijection in an explicit and easy to use form.

Put

$$(3.1) \quad k = \begin{cases} \mathbb{Q}(\sqrt{n}), & \text{if } n \text{ is not the square of an integer;} \\ \mathbb{Q} \oplus \mathbb{Q}, & \text{otherwise.} \end{cases}$$

For integers $b, c \geq 1$, set

$$S_k(b, c) = \{(\mathfrak{a}, \beta) \mid \mathfrak{a} \text{ is an invertible } \mathcal{O}_{k,c}\text{-ideal, } \beta \in \mathfrak{a}^3 \text{ and } \mathbf{N}(\beta \mathfrak{a}^{-3}) = b\}.$$

We call two pairs $(\mathfrak{a}_1, \beta_1)$ and $(\mathfrak{a}_2, \beta_2)$ of $S_k(b, c)$ equivalent, and write $(\mathfrak{a}_1, \beta_1) \sim (\mathfrak{a}_2, \beta_2)$, if there exists an element $\rho \in k^*$ such that $\rho \mathfrak{a}_1 = \mathfrak{a}_2$ and $\rho^3 \beta_1 = \beta_2$. We

write $[\mathfrak{a}, \beta]$ for the equivalence class of (\mathfrak{a}, β) and put

$$S(n) = \bigcup_{(bc)^2 = n/\Delta_k} S_k(b, c) / \sim \quad (\text{disjoint}).$$

We now define a map Ψ_n from $S(n)$ to $\Gamma \backslash L^\vee(n)$ as follows. To each $(\mathfrak{a}, \beta) \in S_k(b, c)$ with $(bc)^2 \Delta_k = n$, choose a \mathbb{Z} -basis α_1, α_2 of \mathfrak{a} with

$$\mathrm{N}_{k/\mathbb{Q}}(\beta)(\alpha_1 \alpha_2^\tau - \alpha_2 \alpha_1^\tau) / \sqrt{\Delta_k} < 0$$

and put

$$\begin{aligned} x(u, v) &= \frac{1}{c\sqrt{\Delta_k} \mathbf{N}\mathfrak{a}^3} (\beta(\alpha_1^\tau u + \alpha_2^\tau v)^3 - \beta^\tau(\alpha_1 u + \alpha_2 v)^3) \\ &= \mathrm{Tr}_{k/\mathbb{Q}} \left(\frac{\beta}{c\sqrt{\Delta_k} \mathbf{N}\mathfrak{a}^3} (\alpha_1^\tau u + \alpha_2^\tau v)^3 \right). \end{aligned}$$

We claim that $x(u, v) \in L^\vee(n)$ and the Γ -orbit of $x(u, v)$ depends only on $[\mathfrak{a}, \beta]$. The condition $\beta \in \mathfrak{a}^3$ is equivalent to $\beta(\mathfrak{a}^\tau)^3 \subseteq \mathbf{N}\mathfrak{a}^3 \mathcal{O}_{k,c} = c\sqrt{\Delta_k} \mathbf{N}\mathfrak{a}^3 \mathcal{O}_{k,c}^\vee$, or $x(u, v) \in L^\vee$. Write $x(u, v) = w((u, v)g)$ with

$$(3.2) \quad w(u, v) = \frac{1}{c\sqrt{\Delta_k} \mathbf{N}\mathfrak{a}^3} (\beta u^3 - \beta^\tau v^3) \quad \text{and} \quad g = \begin{bmatrix} \alpha_1^\tau & \alpha_1 \\ \alpha_2^\tau & \alpha_2 \end{bmatrix},$$

we see that $\mathrm{disc}(x) = (\det g)^6 \mathrm{disc}(w) = -27(bc)^2 \Delta_k$. Moreover, if $(\mathfrak{a}_1, \beta_1)$ is another representative for $[\mathfrak{a}, \beta]$, say $\mathfrak{a}_1 = \rho \mathfrak{a}$ and $\beta_1 = \rho^3 \beta$ for some $\rho \in k^*$, and ω_1, ω_2 is a \mathbb{Z} -basis of \mathfrak{a}_1 with $(\omega_1 \omega_2^\tau - \omega_2 \omega_1^\tau) \mathrm{N}_{k/\mathbb{Q}}(\beta_1) / \sqrt{\Delta_k} < 0$. Then there exists a $\gamma \in \Gamma$ such that $(\omega_1, \omega_2) = (\rho \alpha_1, \rho \alpha_2) \gamma$, and so

$$\mathrm{Tr}_{k/\mathbb{Q}} \left(\frac{\beta_1}{c\sqrt{\Delta_k} \mathbf{N}\mathfrak{a}_1^3} (\omega_1^\tau u + \omega_2^\tau v)^3 \right) = \pm x((u, v)\gamma^T)$$

is in the Γ -orbit of $x(u, v)$. We let Ψ_n be the map sending the equivalence class $[\mathfrak{a}, \beta]$ to the Γ -orbit of $x(u, v)$.

Theorem 3.1. *The map $\Psi_n : S(n) \rightarrow \Gamma \backslash L^\vee(n)$ is a bijection.*

Proof. We prove the surjectivity of Ψ_n by using a refined syzygy identity. Let k denote the quadratic étale algebra defined by (3.1) and let d be the positive integer satisfying $d^2 \Delta_k = n$. Take any $x(u, v) = x_0 u^3 + 3x_1 u^2 v + 3x_2 uv^2 + x_3 v^3 \in L^\vee(n)$. Let

$$(3.3) \quad H(u, v) = -\frac{1}{36} \begin{vmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 x}{\partial u \partial v} \\ \frac{\partial^2 x}{\partial v \partial u} & \frac{\partial^2 x}{\partial v^2} \end{vmatrix} \quad \text{and} \quad J(u, v) = \frac{1}{3} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} \end{vmatrix}$$

denote respectively the Hessian and the Jacobian covariant form of $x(u, v)$. Thus $H(u, v) = B_0 u^2 + B_1 uv + B_2 v^2$, where

$$B_0 = x_1^2 - x_0 x_2, \quad B_1 = x_1 x_2 - x_0 x_3, \quad B_2 = x_2^2 - x_1 x_3,$$

and $J(u, v) = C_0 u^3 + 3C_1 u^2 v + 3C_2 uv^2 + C_3 v^3$, where

$$\begin{aligned} C_0 &= x_0 B_1 - 2x_1 B_0, & C_1 &= -x_1 B_1 + 2x_0 B_2, \\ C_2 &= x_2 B_1 - 2x_3 B_0, & C_3 &= -x_3 B_1 + 2x_2 B_2. \end{aligned}$$

Apply suitable Γ -action on $x(u, v)$ if necessary, we may assume that $B_0 \neq 0$. These covariant forms satisfy the syzygy relation

$$J^2 - \mathrm{disc}(H) x^2 = 4H^3$$

with $\text{disc}(H) = -\text{disc}(x)/27 = n$. What we need is the following refined identity, which can be checked straightforwardly:

$$(3.4) \quad J + d\sqrt{\Delta_k}x = (C_0 + d\sqrt{\Delta_k}x_0)\left(u + \frac{B_1 - d\sqrt{\Delta_k}}{2B_0}v\right)^3.$$

Following [11] we let $b = \gcd(B_0, B_1, B_2)$ denote the content of $H(u, v)$ and put

$$h(u, v) = \frac{1}{b}H(u, v) = b_0u^2 + b_1uv + b_2v^2.$$

Then $\text{disc}(h) = c^2\Delta_k$ for some positive integer c with $bc = d$. Put

$$\alpha = \frac{b_1 + c\sqrt{\Delta_k}}{2b_0} \quad \text{and} \quad \beta = \frac{C_0 + d\sqrt{\Delta_k}x_0}{2bb_0^3}.$$

It is easy to check, for example using [8, Thm 5.12], that $\mathfrak{a} = \mathbb{Z}\mathbf{1} + \mathbb{Z}\alpha$ is an invertible fractional ideal of $\mathcal{O}_{k,c} = \mathbb{Z}\mathbf{1} + \mathbb{Z}b_0\alpha$ with $\mathfrak{a}^{-1} = \mathbb{Z}b_0\mathbf{1} + \mathbb{Z}(b_0\alpha + b_1\mathbf{1})$. Moreover, we have $\mathbf{N}\mathfrak{a} = |b_0|^{-1}$ and $\mathbf{N}_{k/\mathbb{Q}}(\beta) = b/b_0^3$. Thus $\mathbf{N}(\beta\mathfrak{a}^{-3}) = b$ and $(\alpha^\tau - \alpha)\mathbf{N}_{k/\mathbb{Q}}(\beta)/\sqrt{\Delta_k} < 0$. Now (3.4) can be written as

$$J + d\sqrt{\Delta_k}x = 2bb_0^3\beta(u + \alpha^\tau v)^3.$$

So we have

$$x(u, v) = \text{Tr}_{k/\mathbb{Q}}\left(\frac{\beta}{c\sqrt{\Delta_k}\mathbf{N}\mathfrak{a}^3}(u + \alpha^\tau v)^3\right).$$

The condition $x(u, v) \in L^\vee$ implies that $\beta \in \mathfrak{a}^3$. Hence $(\mathfrak{a}, \beta) \in S_k(b, c)$ and the map Ψ_n sends $[\mathfrak{a}, \beta]$ to the Γ -orbit of $x(u, v)$.

The injectivity of Ψ_n follows from the covariance property of $H(u, v)$ and $J(u, v)$. Suppose Ψ_n maps the equivalence class of $(\mathfrak{a}, \beta) \in S_k(b, c)$ to the Γ -orbit of $x(u, v) \in L^\vee(n)$. Without loss of generality, we assume \mathfrak{a} has a \mathbb{Z} -basis α_1, α_2 with $(\alpha_1\alpha_2^\tau - \alpha_2\alpha_1^\tau)\mathbf{N}_{k/\mathbb{Q}}(\beta)/\sqrt{\Delta_k} < 0$ such that

$$\text{Tr}_{k/\mathbb{Q}}\left(\frac{\beta}{c\sqrt{\Delta_k}\mathbf{N}\mathfrak{a}^3}(\alpha_1^\tau u + \alpha_2^\tau v)^3\right) = x(u, v).$$

Let $w(u, v)$ and g be defined by (3.2). Since the Hessian form of $w(u, v)$ is

$$H_w(u, v) = \text{sgn}(\mathbf{N}_{k/\mathbb{Q}}(\beta))\frac{b^3}{n\mathbf{N}\mathfrak{a}^3}uv,$$

the Hessian form of $x(u, v) = w((u, v)g)$ is given by

$$H_x(u, v) = (\det g)^2 H_w((u, v)g) = \text{sgn}(\mathbf{N}_{k/\mathbb{Q}}(\beta))\frac{b}{\mathbf{N}\mathfrak{a}}\mathbf{N}(\alpha_1 u + \alpha_2 v).$$

Hence b is the content of $H_x(u, v)$. Consequently, b and c are uniquely determined by the Γ -orbit of $x(u, v)$. Similarly, using the covariance property of the Jacobian form $J_x(u, v)$ of $x(u, v)$, we get

$$\frac{1}{b}J_x(u, v) + c\sqrt{\Delta_k}x(u, v) = \frac{2}{\mathbf{N}\mathfrak{a}^3}\beta(\alpha_1^\tau u + \alpha_2^\tau v)^3.$$

From this we deduce that $[\mathfrak{a}, \beta]$ is uniquely determined by the Γ -orbit of $x(u, v)$. \square

Transition to L -series of quadratic orders. For each pair of positive integers b and c , we put

$$\Omega_k(b, c) = \{ \text{integral invertible } \mathcal{O}_{k,c}\text{-ideal } \mathfrak{b} \mid \mathbf{N}\mathfrak{b} = b, [\mathfrak{b}] \in Cl_{k,c}^3 \}.$$

Take any $(\mathfrak{a}, \beta) \in S_k(b, c)$. Clearly $\beta \mathfrak{a}^{-3}$ is an integral ideal of $\mathcal{O}_{k,c}$ contained in $\Omega_k(b, c)$ which depends only on the equivalence class of (\mathfrak{a}, β) . We let Φ denote the map from $S_k(b, c)/\sim$ to $\Omega_k(b, c)$ sending $[\mathfrak{a}, \beta]$ to $\beta \mathfrak{a}^{-3}$. Furthermore, put

$$Cl_{k,c}^{(3)} = \{ [\mathfrak{a}] \in Cl_{k,c} \mid [\mathfrak{a}]^3 = 1 \}.$$

Lemma 3.2. *Under the map Φ , each integral $\mathcal{O}_{k,c}$ -ideal in $\Omega_k(b, c)$ has exactly $|Cl_{k,c}^{(3)}| |\mathcal{O}_{k,c}^* / \mathcal{O}_{k,c}^{*3}|$ inverse images.*

Proof. By definition, for each $\mathfrak{b} \in \Omega_k(b, c)$, there exists an invertible $\mathcal{O}_{k,c}$ -ideal \mathfrak{a} and an element $\beta \in k^*$ with $(\mathfrak{a}, \beta) \in S_k(b, c)$ such that $\mathfrak{b} = \beta \mathfrak{a}^{-3}$. Furthermore, if $(\mathfrak{a}_1, \beta_1) \in S_k(b, c)$ is a pair satisfying $\beta_1 \mathfrak{a}_1^{-3} = \mathfrak{b}$, then $[\mathfrak{a}_1 / \mathfrak{a}]^3 = 1$. So we have $|Cl_{k,c}^{(3)}|$ choices for the ideal class of \mathfrak{a}_1 . Once an ideal class is chosen and a representative \mathfrak{a}_1 is fixed, β_1 is determined by \mathfrak{b} up to a unit in $\mathcal{O}_{k,c}$, hence the equivalence class $[\mathfrak{a}_1, \beta_1]$ is determined by \mathfrak{b} up to $|\mathcal{O}_{k,c}^* / \mathcal{O}_{k,c}^{*3}|$ possibilities. \square

Theorem 3.3. *With the notation above, we have*

$$\xi_1^\vee(s) = \sum_{k \text{ imaginary quadratic}} \frac{1}{|\Delta_k|^s} \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \sum_{\chi \in Cl_{k,c}^\vee, \chi^3=1} L(2s, \chi, \mathcal{O}_{k,c})$$

and

$$\begin{aligned} \xi_2^\vee(s) = & \sum_{k \text{ real quadratic}} \frac{3}{|\Delta_k|^s} \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \sum_{\chi \in Cl_{k,c}^\vee, \chi^3=1} L(2s, \chi, \mathcal{O}_{k,c}) \\ & + \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \sum_{\chi \in Cl_{\mathbb{Q}^2,c}^\vee, \chi^3=1} L(2s, \chi, \mathcal{O}_{\mathbb{Q}^2,c}), \end{aligned}$$

where in the inner sums above χ goes through all characters of $Cl_{k,c}$ (and $Cl_{\mathbb{Q}^2,c}$) satisfying $\chi^3 = 1$.

Proof. Let $x(u, v) \in L^\vee$ and let $[\mathfrak{a}, \beta] \in S_k(b, c)/\sim$ be its equivalence class under the bijection Ψ_n . It is easy to check, using ([17, Prop 2.12], [11]), that $|\Gamma_x| = 3$ if and only if $k = \mathbb{Q}(\sqrt{-3})$ and $c = 1$. Thus

$$|\mathcal{O}_{k,c}^* / \mathcal{O}_{k,c}^{*3}| = \begin{cases} 3 |\Gamma_x|, & \text{if } k \text{ is a real quadratic field,} \\ |\Gamma_x|, & \text{otherwise.} \end{cases}$$

Let $\Omega_k(c) = \bigcup_{b \geq 1} \Omega_k(b, c)$. Putting Theorem 3.1 and Lemma 3.2 together, we see that each $\mathfrak{b} \in \Omega_k(c)$ corresponds to $|\mathcal{O}_{k,c}^* / \mathcal{O}_{k,c}^{*3}| |Cl_{k,c}^{(3)}|$ Γ -orbits in L^\vee and all these orbits have the discriminant $-27\Delta_k c^2 \mathbf{N}\mathfrak{b}^2$. Thus

$$\begin{aligned} \xi_1^\vee(s) &= \sum_{x \in \Gamma \backslash L_+^\vee} \frac{1}{|\Gamma_x|} |\text{disc}(x)/27|^{-s} \\ &= \sum_{k \text{ imaginary quadratic}} \frac{1}{|\Delta_k|^s} \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \sum_{\mathfrak{b} \in \Omega_k(c)} \frac{|Cl_{k,c}^{(3)}|}{\mathbf{N}\mathfrak{b}^{2s}}. \end{aligned}$$

The inner most sum can be expressed as a summation of $L(2s, \chi, \mathcal{O}_{k,c})$ over all characters χ of $Cl_{k,c}$ satisfying $\chi^3 = 1$. This proves the first part of Theorem 3.3. Similarly, we have

$$\begin{aligned} \xi_2^\vee(s) &= \sum_{k \text{ real quadratic}} \frac{3}{|\Delta_k|^s} \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \sum_{\mathfrak{b} \in \Omega_k(c)} \frac{|Cl_{k,c}^{(3)}|}{\mathbf{N}\mathfrak{b}^{2s}} \\ &\quad + \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \sum_{\mathfrak{b} \in \Omega_{\mathbb{Q}^2}(c)} \frac{|Cl_{\mathbb{Q}^2,c}^{(3)}|}{\mathbf{N}\mathfrak{b}^{2s}}. \end{aligned}$$

Converting to L -series of orders, we obtain the second part of Theorem 3.3. \square

4. TRANSFORMING $\xi_i(s)$ INTO TRUNCATED L -SERIES OF ORDERS

In this section we express Datskovsky & Wright's beautiful Theorem [5] in terms of truncated L -series of quadratic orders.

Theorem 4.1 (Datskovsky & Wright). *The Shintani zeta functions $\xi_i(s)$ ($i = 1, 2$) can be expressed as*

$$(4.1) \quad \frac{\xi_1(s)}{\zeta(4s)\zeta(6s-1)} = 2I + II + \frac{2}{3}III + \frac{1}{3}IV,$$

$$(4.2) \quad \frac{\xi_2(s)}{\zeta(4s)\zeta(6s-1)} = 2I' + II',$$

where

$$\begin{aligned} I &= \sum_{K \text{ non-Galois cubic field with } \Delta_K > 0} |\Delta_K|^{-s} \frac{\zeta_K(2s)}{\zeta_K(4s)}, \\ II &= \sum_{k \text{ real quadratic field}} |\Delta_k|^{-s} \frac{\zeta(2s)}{\zeta(4s)} \frac{\zeta_k(2s)}{\zeta_k(4s)}, \\ III &= \sum_{K \text{ cyclic cubic field}} |\Delta_K|^{-s} \frac{\zeta_K(2s)}{\zeta_K(4s)}, \\ IV &= \left(\frac{\zeta(2s)}{\zeta(4s)} \right)^3 \end{aligned}$$

and

$$\begin{aligned} I' &= \sum_{K \text{ complex cubic field}} |\Delta_K|^{-s} \frac{\zeta_K(2s)}{\zeta_K(4s)}, \\ II' &= \sum_{k \text{ imaginary quadratic field}} |\Delta_k|^{-s} \frac{\zeta(2s)}{\zeta(4s)} \frac{\zeta_k(2s)}{\zeta_k(4s)}. \end{aligned}$$

In the expressions above conjugate cubic fields are counted only once.

Remark 4.2. Let $E(3)$ denote the set of isomorphism classes of étale algebras of dimension 3 over \mathbb{Q} . Let $\text{Aut}(A)$ denote the automorphic group of an étale algebra A and $\zeta_A(s)$ the zeta function of A . Then Datskovsky & Wright's Theorem can

be succinctly expressed as

$$\xi_i(s) = \zeta(4s)\zeta(6s-1) \sum_{\substack{A \in E(3) \\ (-1)^{i-1}\Delta_A > 0}} \frac{2}{|\text{Aut}(A)|} |\Delta_A|^{-s} \frac{\zeta_A(2s)}{\zeta_A(4s)}, \quad i = 1, 2.$$

We shall compute these contributions case by case. Our method differs from that of Nakagawa's in that we bring the whole Artin L -function down to quadratic orders. To do this we need to make his map in Lemma 1.10 of [11] (i.e., the map ϕ' in Lemma 4.3 below) explicit. The following Galois eigenspace argument leading to Lemma 4.3 are taken directly from Nakagawa's work [11], with some notational changes and clarifications, especially on the construction of the map ϕ' .

Descending to quadratic orders: K is a non-Galois cubic field. In this case the Galois closure \tilde{K} of K contains a unique quadratic field k , and \tilde{K}/k is a cyclic cubic extension whose conductor is f , the positive integer satisfying $\Delta_K = f^2\Delta_k$.

Let $A_k(f)$ denote the odd part of the ray class group $Cl_k(f) = I_k(f)/P_{k,1}(f)$. By class field theory, \tilde{K}/k corresponds to a subgroup H of $A_k(f)$ of index 3 whose conductor is f . Let τ be the non-trivial automorphism of the quadratic field k . Then τ acts on $A_k(f)$ and we have the decomposition

$$A_k(f) = \frac{1+\tau}{2}A_k(f) \oplus \frac{1-\tau}{2}A_k(f) = A_k^+(f) \oplus A_k^-(f),$$

where $A_k^+(f) = \{a \in A_k(f) \mid a^\tau = a\}$ and $A_k^-(f) = \{a \in A_k(f) \mid a^\tau = a^{-1}\}$. Since \tilde{K}/\mathbb{Q} is a Galois extension, H is invariant under the action of τ . Thus we have decomposition $H = H^+ \oplus H^-$ with $H^+ \subseteq A_k^+(f)$ and $H^- \subseteq A_k^-(f)$. Furthermore, since $\text{Gal}(\tilde{K}/\mathbb{Q}) \simeq S_3$, the action of τ on $A_k(f)/H$ is non-trivial. Hence $H^+ = A_k^+(f)$ and $[A_k^-(f) : H^-] = 3$. Conversely, given any subgroup H^- of $A_k^-(f)$ of index 3 and having conductor f , it corresponds to a unique a cyclic cubic extension \tilde{K}/k such that \tilde{K}/\mathbb{Q} is a Galois extension with $\text{Gal}(\tilde{K}/\mathbb{Q}) \simeq S_3$, and the three conjugate cubic subfields K of \tilde{K} have discriminant $\Delta_K = f^2\Delta_k$.

Write $Cl_{k,f}$ for the class group of $\mathcal{O}_{k,f}$ as before. Let $\phi : I_k(f) \rightarrow I(\mathcal{O}_{k,f}, f)$ be the isomorphism given in Corollary 2.2. Then

$$P_{k,\mathbb{Z}}(f) = \{\alpha\mathcal{O}_k \mid \alpha \in k, \alpha \equiv a \pmod{*f} \text{ for some } a \in \mathbb{Z} \text{ with } (a, f) = 1\}$$

is the inverse image of $P(\mathcal{O}_{k,f}, f)$ under ϕ . By Lemma 2.3, we have

$$Cl_{k,f} \simeq I(\mathcal{O}_{k,f}, f)/P(\mathcal{O}_{k,f}, f) \simeq I_k(f)/P_{k,\mathbb{Z}}(f).$$

Thus ϕ induces a homomorphism $\tilde{\phi}$ from the ray class group $Cl_k(f)$ onto $Cl_{k,f}$ which extends to the exact sequence of $\text{Gal}(k/\mathbb{Q})$ -modules :

$$(4.3) \quad 1 \longrightarrow P_{k,\mathbb{Z}}(f)/P_{k,1}(f) \longrightarrow Cl_k(f) \xrightarrow{\tilde{\phi}} Cl_{k,f} \longrightarrow 1.$$

Let $C_k(f)$ denote the odd part of $Cl_{k,f}$. Notice that $a^\tau = a^{-1}$ for all $a \in C_k(f)$. By restricting the exact sequence (4.3) first to the odd part of groups, and then to the minus part of the decomposition induced by τ , and using the fact that the minus part of $P_{k,\mathbb{Z}}(f)/P_{k,1}(f)$ vanishes, Nakagawa obtained the following result:

Lemma 4.3. *The restriction ϕ' of $\tilde{\phi}$ to the direct summand $A_k^-(f)$ of $Cl_k(f)$ is an isomorphism between $A_k^-(f)$ and $C_k(f)$.*

Now comes our real innovation of the paper. Let χ be a non-trivial character of $\text{Gal}(\tilde{K}/k)$. By induction formula and additivity of Artin L -function, we have

$$\zeta_K(s) = \zeta(s) L(s, \chi, \tilde{K}/k).$$

Through Artin map, we may regard χ as a primitive character of the ray class group $Cl_k(f)$ and write the Artin L -function as

$$L(s, \chi, \tilde{K}/k) = \prod_{\tilde{\mathfrak{p}} \nmid f} \left(1 - \frac{\chi(\tilde{\mathfrak{p}})}{\mathbf{N}\tilde{\mathfrak{p}}^s} \right)^{-1},$$

where the product is over primes $\tilde{\mathfrak{p}}$ of \mathcal{O}_k not dividing f . From the discussion above, we know that χ is actually a character of the direct summand $A_k^-(f)$ of $Cl_k(f)$, so we may further identify χ through the isomorphism ϕ' as a cubic character of $Cl_{k,f}$. It is easy to see that χ is also primitive in the sense of Lemma 2.4. Under these identifications, we have $\chi(\tilde{\mathfrak{a}}) = \chi(\tilde{\mathfrak{a}} \cap \mathcal{O}_{k,f})$ for all integral ideals $\tilde{\mathfrak{a}}$ of \mathcal{O}_k prime to f . Thus

$$(4.4) \quad L(s, \chi, \tilde{K}/k) = \prod_{\mathfrak{p} \nmid f} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s} \right)^{-1} = L^*(s, \chi, \mathcal{O}_{k,f}),$$

where in the product \mathfrak{p} goes over all primes of $\mathcal{O}_{k,f}$ not dividing f .

Since each pair of complex conjugate primitive cubic characters $\chi, \bar{\chi}$ of $Cl_{k,f}$ corresponds bijectively through ϕ' with an isomorphic class of non-abelian cubic field K whose Galois closure \tilde{K} is a cyclic cubic extension of k of conductor f , we can write

$$2I = \sum_{k \text{ real quadratic}} \frac{1}{|\Delta_k|^s} \sum_{f=1}^{\infty} f^{-2s} \sum_{\substack{\chi \in Cl_{k,f}^\vee \text{ primitive} \\ \chi^3=1, \chi \neq 1}} \frac{\zeta(2s)}{\zeta(4s)} \frac{L^*(2s, \chi, \mathcal{O}_{k,f})}{L^*(4s, \chi, \mathcal{O}_{k,f})}.$$

If we allow χ in the inner sum take the trivial primitive character of $Cl_{k,f}$ (which is possible only when $f=1$), we need to add the sum

$$\sum_{k \text{ real quadratic}} \frac{1}{|\Delta_k|^s} \frac{\zeta(2s)}{\zeta(4s)} \frac{\zeta_k(2s)}{\zeta_k(4s)}.$$

But this is exactly the contribution coming from part *II*. Thus we have

$$2I + II = \sum_{k \text{ real quadratic}} |\Delta_k|^{-s} \sum_{f=1}^{\infty} f^{-2s} \sum_{\substack{\chi \in Cl_{k,f}^\vee \text{ primitive} \\ \chi^3=1}} \frac{\zeta(2s)}{\zeta(4s)} \frac{L^*(2s, \chi, \mathcal{O}_{k,f})}{L^*(4s, \chi, \mathcal{O}_{k,f})}.$$

Apply the same argument to I' and II' , we get

$$2I' + II' = \sum_{k \text{ imaginary quadratic}} |\Delta_k|^{-s} \sum_{f=1}^{\infty} f^{-2s} \sum_{\substack{\chi \in Cl_{k,f}^\vee \text{ primitive} \\ \chi^3=1}} \frac{\zeta(2s)}{\zeta(4s)} \frac{L^*(2s, \chi, \mathcal{O}_{k,f})}{L^*(4s, \chi, \mathcal{O}_{k,f})}.$$

Descending to quadratic orders: K is a cyclic cubic field. In this case the conductor of K/\mathbb{Q} is a positive integer satisfying $f^2 = \Delta_K$. By class field theory, cyclic cubic extensions K of \mathbb{Q} of conductor f correspond bijectively with pairs of primitive cubic characters χ_1, χ_1^{-1} of the ray class group

$$Cl_{\mathbb{Q}}(f) = I_{\mathbb{Q}}(f)/P_{\mathbb{Q},1}(f) \simeq (\mathbb{Z}/f\mathbb{Z})^*/\{\pm 1\}.$$

Identify χ_1 and χ_1^{-1} with characters of $\text{Gal}(K/\mathbb{Q})$ through the Artin map, and use the additivity of Artin L -functions, we obtain

$$\zeta_K(s) = \zeta(s) L(s, \chi_1) L(s, \chi_1^{-1}),$$

where

$$L(s, \chi_1) = \prod_{p \nmid f} (1 - \chi_1(p) p^{-s})^{-1}.$$

It can be checked directly, or using Corollary A.5, that there is a bijection between the characters χ_1 of $Cl_{\mathbb{Q}}(f)$ and the characters χ of $Cl_{\mathbb{Q}^2, f}$ such that χ_1 is primitive if and only if χ is primitive. The bijection is given by $\chi(\mathfrak{a}) = \chi_1(a_1/a_0)$, where we put $\mathfrak{a} \mathcal{O}_{\mathbb{Q}^2} = (a_0\mathbb{Z}, a_1\mathbb{Z})$ with $a_0, a_1 \in \mathbb{Z}^+$, for all integral ideals \mathfrak{a} of $\mathcal{O}_{\mathbb{Q}^2, f}$ prime to $f \mathcal{O}_{\mathbb{Q}^2}$. Furthermore, we have

$$L(s, \chi_1) L(s, \chi_1^{-1}) = L^*(s, \chi, \mathcal{O}_{\mathbb{Q}^2, f}).$$

Hence

$$2 III = \sum_{f=1}^{\infty} \frac{1}{f^{2s}} \sum_{\substack{\chi \in Cl_{\mathbb{Q}^2, f}^{\vee} \\ \chi^3=1, \chi \neq 1}} \frac{\zeta(2s)}{\zeta(4s)} \frac{L^*(2s, \chi, \mathcal{O}_{\mathbb{Q}^2, f})}{L^*(4s, \chi, \mathcal{O}_{\mathbb{Q}^2, f})}.$$

Again, if we allow χ to take the trivial primitive character of $Cl_{\mathbb{Q}}(f)$ in the inner sum (which is possible only when $f=1$), we add precisely the term

$$IV = \left(\frac{\zeta(2s)}{\zeta(4s)} \right)^3.$$

Thus

$$2 III + IV = \sum_{f=1}^{\infty} \frac{1}{f^{2s}} \sum_{\substack{\chi \in Cl_{\mathbb{Q}^2, f}^{\vee} \\ \chi^3=1}} \frac{\zeta(2s)}{\zeta(4s)} \frac{L^*(2s, \chi, \mathcal{O}_{\mathbb{Q}^2, f})}{L^*(4s, \chi, \mathcal{O}_{\mathbb{Q}^2, f})}.$$

Substitute the formulae above back to Theorem 4.1, we obtain:

Theorem 4.4.

$$\begin{aligned} \frac{\xi_2(s)}{\zeta(2s)\zeta(6s-1)} &= \sum_{k \text{ imaginary quadratic}} |\Delta_k|^{-s} \sum_{f=1}^{\infty} f^{-2s} \sum_{\substack{\chi \in Cl_{k, f}^{\vee} \\ \chi^3=1}} \frac{L^*(2s, \chi, \mathcal{O}_{k, f})}{L^*(4s, \chi, \mathcal{O}_{k, f})}, \\ \frac{\xi_1(s)}{\zeta(2s)\zeta(6s-1)} &= \sum_{k \text{ real quadratic}} |\Delta_k|^{-s} \sum_{f=1}^{\infty} f^{-2s} \sum_{\substack{\chi \in Cl_{k, f}^{\vee} \\ \chi^3=1}} \frac{L^*(2s, \chi, \mathcal{O}_{k, f})}{L^*(4s, \chi, \mathcal{O}_{k, f})} \\ &\quad + \frac{1}{3} \sum_{f=1}^{\infty} f^{-2s} \sum_{\substack{\chi \in Cl_{\mathbb{Q}^2, f}^{\vee} \\ \chi^3=1}} \frac{L^*(2s, \chi, \mathcal{O}_{\mathbb{Q}^2, f})}{L^*(4s, \chi, \mathcal{O}_{\mathbb{Q}^2, f})}. \end{aligned}$$

5. RELATIONS BETWEEN $L^*(s, \chi, \mathcal{O}_{k,m})$ AND $L(s, \chi, \mathcal{O}_{k,m})$

In this section we adopt the following convention. Let k be a quadratic étale algebra. We define the quadratic order $\mathcal{O}_{k,m}$ and its class group $Cl_{k,m}$ as before. Let χ be a character of $Cl_{k,m}$. For economy of notations, we denote all characters induced from χ by the same symbol χ . Similarly, if χ is defined at $Cl_{k,m'}$ for some $m' | m$, then we still use χ to denote the character of $Cl_{k,m'}$ which induces χ .

The goal of this section is to prove the following identity :

Theorem 5.1. *Let k be a quadratic étale algebra and let χ be a primitive character of $Cl_{k,f}$ of odd order, then*

$$(5.1) \quad \sum_{d=1}^{\infty} \frac{1}{d^s} L(s, \chi, \mathcal{O}_{k,fd}) = \zeta(s) \zeta(3s-1) \frac{L^*(s, \chi, \mathcal{O}_{k,f})}{L^*(2s, \chi, \mathcal{O}_{k,f})}.$$

Our main result Theorem 1.1 follows easily from this identity. By Lemma 2.4, each character χ of $Cl_{k,c}$ is induced by a unique primitive character at $Cl_{k,f}$, where f is the conductor of χ . Group the characters according to their primitive characters, and apply Theorem 5.1, we get

$$\begin{aligned} & \sum_{c=1}^{\infty} \frac{1}{c^s} \sum_{\substack{\chi \in Cl_{k,c}^{\vee} \\ \chi^3=1}} L(s, \chi, \mathcal{O}_{k,c}) \\ &= \sum_{f=1}^{\infty} \frac{1}{f^s} \sum_{\substack{\chi \in Cl_{k,f}^{\vee} \text{ primitive} \\ \chi^3=1}} \sum_{d=1}^{\infty} \frac{1}{d^s} L(s, \chi, \mathcal{O}_{k,fd}) \\ &= \zeta(s) \zeta(3s-1) \sum_{f=1}^{\infty} \frac{1}{f^s} \sum_{\substack{\chi \in Cl_{k,f}^{\vee} \text{ primitive} \\ \chi^3=1}} \frac{L^*(s, \chi, \mathcal{O}_{k,f})}{L^*(2s, \chi, \mathcal{O}_{k,f})}. \end{aligned}$$

Take the sum over all quadratic étale algebras k with positive (resp., negative) discriminants, and apply Theorem 3.3 and Theorem 4.4, we obtain Theorem 1.1.

From L-series of orders to truncated L-series. To compute $L(s, \chi, \mathcal{O}_{k,m})$, we have to deal with integral $\mathcal{O}_{k,m}$ -ideals not necessarily prime to the conductor. The following Lemma is crucial.

Lemma 5.2. *Let \mathfrak{a} be an integral ideal of $\mathcal{O}_{k,m}$ and let c be the smallest positive integer contained in $\mathfrak{a} + m\mathcal{O}_k$. Put $m' = m/c \in \mathbb{Z}$. Then $\mathfrak{a} + m\mathcal{O}_k = c\mathcal{O}_{k,m'}$ and $c^{-1}\mathfrak{a}\mathcal{O}_{k,m'}$ is an integral $\mathcal{O}_{k,m'}$ -ideal prime to the conductor of $\mathcal{O}_{k,m'}$.*

Proof. By modular law, we have

$$\begin{aligned} \mathfrak{a} + m\mathcal{O}_k &= (\mathfrak{a} + m\mathcal{O}_k) \cap (\mathbb{Z}\mathbf{1} + m\mathcal{O}_k) \\ &= c\mathbb{Z}\mathbf{1} + m\mathcal{O}_k \\ &= c\mathcal{O}_{k,m'}. \end{aligned}$$

Thus $c^{-1}\mathfrak{a}\mathcal{O}_{k,m'} + m'\mathcal{O}_k = \mathcal{O}_{k,m'}$. □

For $m, m' \in \mathbb{Z}^+$ with $m' | m$, let $\varphi_{k,m',m} : Cl_{k,m} \rightarrow Cl_{k,m'}$ denote the surjective homomorphism sending the ideal class of \mathfrak{a} to the ideal class of $\mathfrak{a}\mathcal{O}_{k,m'}$, for every invertible $\mathcal{O}_{k,m}$ -ideal \mathfrak{a} .

Lemma 5.3. *For every ideal class \mathfrak{A} of $Cl_{k,m}$, we have*

$$(5.2) \quad \zeta(s, \mathfrak{A}, \mathcal{O}_{k,m}) = \sum_{m' | m} \frac{[\mathcal{O}_{k,m'}^* : \mathcal{O}_{k,m}^*]}{(m/m')^{2s}} \zeta^*(s, \varphi_{k,m',m}(\mathfrak{A}), \mathcal{O}_{k,m'}).$$

Proof. Let \mathfrak{a} be an integral invertible $\mathcal{O}_{k,m}$ -ideal belonging to \mathfrak{A} . Let c and m' be positive integers such that $(\mathfrak{a} + m\mathcal{O}_k) \cap \mathbb{Z}\mathbf{1} = c\mathbb{Z}\mathbf{1}$ and $m' = m/c$. Then Lemma 5.2 implies that $\mathfrak{b} = c^{-1}\mathfrak{a}\mathcal{O}_{k,m'} \in \varphi_{k,m',m}(\mathfrak{A})$ is an integral ideal of $\mathcal{O}_{k,m'}$ prime to $m'\mathcal{O}_k$. Conversely, given integers c and m' with $cm' = m$, and given integral $\mathcal{O}_{k,m'}$ -ideals $\mathfrak{b} \in \varphi_{k,m',m}(\mathfrak{A})$ prime to $m'\mathcal{O}_k$, all the invertible $\mathcal{O}_{k,m}$ -ideals \mathfrak{a} under $c\mathfrak{b}$ are integral $\mathcal{O}_{k,m}$ -ideals, as $c\mathfrak{b} \subseteq c\mathcal{O}_{k,m'} \subseteq \mathcal{O}_{k,m}$. Moreover, all these $\mathcal{O}_{k,m}$ -ideals \mathfrak{a} satisfy

$$\mathcal{O}_k(\mathfrak{a} + m\mathcal{O}_k) = c\mathcal{O}_k(\mathfrak{b} + m'\mathcal{O}_k) = c\mathcal{O}_k,$$

hence $(\mathfrak{a} + m\mathcal{O}_k) \cap \mathbb{Z}\mathbf{1} = c\mathbb{Z}\mathbf{1}$ by Lemma 5.2. It is easy to see that under each $\mathcal{O}_{k,m'}$ -ideal $m'\mathfrak{b}$, there are precisely $[\mathcal{O}_{k,m'}^* : \mathcal{O}_{k,m}^*]$ invertible $\mathcal{O}_{k,m}$ -ideals lying in the class \mathfrak{A} . The Lemma now follows by grouping the terms in $\zeta(s, \mathfrak{A}, \mathcal{O}_{k,m})$ according to the smallest positive integer contained in $\mathfrak{a} + m\mathcal{O}_k$. \square

Theorem 5.4. *Let χ be a character of $Cl_{k,m}$ of conductor f , then*

$$L(s, \chi, \mathcal{O}_{k,m}) = \sum_{f | m' | m} \frac{|U(\mathcal{O}_{k,m'}, \mathcal{O}_{k,m})|}{(m/m')^{2s}} L^*(s, \chi, \mathcal{O}_{k,m'}).$$

In particular, if χ is a primitive character of $Cl_{k,m}$, then we have

$$L(s, \chi, \mathcal{O}_{k,m}) = L^*(s, \chi, \mathcal{O}_{k,m}).$$

Proof. By Lemma 5.3, we have

$$\begin{aligned} L(s, \chi, \mathcal{O}_{k,m}) &= \sum_{\mathfrak{A} \in Cl_{k,m}} \chi(\mathfrak{A}) \zeta(s, \mathfrak{A}, \mathcal{O}_{k,m}) \\ &= \sum_{m' | m} \frac{[\mathcal{O}_{k,m'}^* : \mathcal{O}_{k,m}^*]}{(m/m')^{2s}} \sum_{\mathfrak{A}' \in Cl_{k,m'}} \left(\sum_{\substack{\mathfrak{A} \in Cl_{k,m} \\ \varphi_{k,m',m}(\mathfrak{A}) = \mathfrak{A}'}} \chi(\mathfrak{A}) \right) \zeta^*(s, \mathfrak{A}', \mathcal{O}_{k,m'}). \end{aligned}$$

Since $\varphi_{k,m',m}$ is surjective, given $\mathfrak{A}' \in Cl_{k,m'}$, there exists a $\mathfrak{A}^* \in Cl_{k,m}$ such that $\varphi_{k,m',m}(\mathfrak{A}^*) = \mathfrak{A}'$. Thus we have

$$\begin{aligned} \sum_{\substack{\mathfrak{A} \in Cl_{k,m} \\ \varphi_{k,m',m}(\mathfrak{A}) = \mathfrak{A}'}} \chi(\mathfrak{A}) &= \chi(\mathfrak{A}^*) \sum_{\mathfrak{B} \in \text{Ker } \varphi_{k,m',m}} \chi(\mathfrak{B}) \\ &= \begin{cases} \chi(\mathfrak{A}^*) |\text{Ker } \varphi_{k,m',m}|, & \text{if } \chi \text{ is trivial on } \text{Ker } \varphi_{k,m',m}; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The sum above is non-zero only if χ is defined at $Cl_{k,m'}$. This happens precisely when m' is a multiple of f . The Theorem follows by applying (2.1). \square

Remark 5.5. Let μ denote the Möbius function on \mathbb{Z} and let χ be a character of $Cl_{k,m}$ of conductor f . Then we have

$$L^*(s, \chi, \mathcal{O}_{k,m}) = \sum_{f | m' | m} \mu(m/m') \frac{|U(\mathcal{O}_{k,m'}, \mathcal{O}_{k,m})|}{(m/m')^{2s}} L(s, \chi, \mathcal{O}_{k,m'}).$$

Theorem 5.4 and the above formula can be vastly generalized to orders of arbitrary étale algebras. However, Lemma 2.4 no longer holds for orders of étale algebras of high degrees.

Proof of Theorem 5.1. We begin with a technical Lemma.

Lemma 5.6. *For every positive integer d , we have*

$$(5.3) \quad \sum_{u=1}^{\infty} \frac{|U(\mathcal{O}_{k,d}, \mathcal{O}_{k,ud})|}{u^s} = \zeta(s-1) \prod_{p \nmid d} \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p^s}\right),$$

where in the case $k = \mathbb{Q}^2$, it is understood that $\left(\frac{\Delta_k}{p}\right) = 1$ for all primes p .

Proof. Apply the class number formula of orders (see [10, p.95]) to (2.1), we get

$$|U(\mathcal{O}_{k,d}, \mathcal{O}_{k,c})| = \frac{c}{d} \prod_{\substack{p|c \\ p \nmid d}} \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p}\right)$$

for every multiple c of d . The formula is also valid when $k = \mathbb{Q}^2$ by defining $\left(\frac{\Delta_k}{p}\right) = 1$ for all p . Write u in the form $u_1 u_2$, where $u_1, u_2 \in \mathbb{Z}^+$ such that all prime factors of u_1 are divisors of d and $(u_2, d) = 1$, the left hand side of (5.3) becomes

$$\begin{aligned} & \sum_{u_1} \frac{1}{u_1^{s-1}} \sum_{u_2} \frac{1}{u_2^{s-1}} \prod_{p|u_2} \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p}\right) \\ &= \prod_{p|d} \left(1 - \frac{1}{p^{s-1}}\right)^{-1} \prod_{p \nmid d} \left(1 + \left(\frac{1}{p^{s-1}} + \frac{1}{p^{2(s-1)}} + \cdots\right) \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p}\right)\right) \end{aligned}$$

which is easily seen equal to the right hand side of (5.3). \square

We now return to the proof of Theorem 5.1. Let χ be a character of the class groups of orders of k of conductor f . Apply Theorem 5.4 and Lemma 5.6 to the left hand side of (5.1), we obtain

$$\begin{aligned} & \sum_{d=1}^{\infty} \frac{1}{d^s} L(s, \chi, \mathcal{O}_{k,fd}) \\ &= \sum_{d=1}^{\infty} \frac{1}{d^s} \sum_{m|d} \frac{|U(\mathcal{O}_{k,fm}, \mathcal{O}_{k,fd})|}{(d/m)^{2s}} L^*(s, \chi, \mathcal{O}_{k,fm}) \\ &= \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{u=1}^{\infty} \frac{|U(\mathcal{O}_{k,fm}, \mathcal{O}_{k,ufm})|}{u^{3s}} L^*(s, \chi, \mathcal{O}_{k,fm}) \\ &= \zeta(3s-1) \sum_{m=1}^{\infty} \frac{1}{m^s} \prod_{p \nmid fm} \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p^{3s}}\right) \prod_{\mathfrak{p}|p} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}, \end{aligned}$$

where in the last product we let \mathfrak{p} run over all prime ideals of $\mathcal{O}_{k,f}$ dividing p , and in the case $k = \mathbb{Q}^2$, we have $\left(\frac{\Delta_k}{p}\right) = 1$ for all p . By bringing those Euler factors

prime to f out of the summation, we get

$$(5.4) \quad \zeta(3s-1) L^*(s, \chi, \mathcal{O}_{k,f}) \prod_{q \nmid f} \left(1 - \left(\frac{\Delta_k}{q}\right) \frac{1}{q^{3s}}\right) \\ \times \sum_{m=1}^{\infty} \frac{1}{m^s} \prod_{\substack{p \mid m \\ p \nmid f}} \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p^{3s}}\right)^{-1} \prod_{\mathfrak{p} \mid p} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s}\right).$$

Write m in the form $m_1 m_2$, where $m_1, m_2 \in \mathbb{Z}^+$ such that all prime factors of m_1 divides f and $(m_2, f) = 1$, the summation part in (5.4) becomes

$$\sum_{m_1} \frac{1}{m_1^s} \sum_{m_2} \frac{1}{m_2^s} \prod_{p \mid m_2} \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p^{3s}}\right)^{-1} \prod_{\mathfrak{p} \mid p} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s}\right) \\ = \prod_{p \mid f} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \nmid f} \left(1 + \frac{1}{p^s - 1} \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p^{3s}}\right)^{-1} \prod_{\mathfrak{p} \mid p} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s}\right)\right) \\ = \zeta(s) \prod_{p \nmid f} \left(1 - \frac{1}{p^s} + \frac{1}{p^s} \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p^{3s}}\right)^{-1} \prod_{\mathfrak{p} \mid p} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s}\right)\right).$$

Substitute back to (5.4), we obtain

$$\sum_{d=1}^{\infty} \frac{1}{d^s} L(s, \chi, \mathcal{O}_{k,fd}) = \zeta(s) \zeta(3s-1) L^*(s, \chi, \mathcal{O}_{k,f}) \\ \times \prod_{p \nmid f} \left(\left(1 - \frac{1}{p^s}\right) \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p^{3s}}\right) + \frac{1}{p^s} \prod_{\mathfrak{p} \mid p} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s}\right) \right).$$

The proof of Theorem 5.1 is now reduced to the following Lemma.

Lemma 5.7. *If χ is a character of $Cl_{k,f}$ of odd order and p is a rational prime not dividing f , then we have*

$$\left(1 - \frac{1}{p^s}\right) \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p^{3s}}\right) + \frac{1}{p^s} \prod_{\mathfrak{p} \mid p} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s}\right) = \prod_{\mathfrak{p} \mid p} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^{2s}}\right),$$

where in the products \mathfrak{p} runs through prime ideals of $\mathcal{O}_{k,f}$ dividing p , and in the case $k = \mathbb{Q}^2$, it is understood that $\left(\frac{\Delta_k}{p}\right) = 1$ for all primes p .

Proof. We claim that for every $p \nmid f$, there exists an $a_p \in \mathbb{C}$ such that

$$(5.5) \quad \prod_{\mathfrak{p} \mid p} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s}\right) = 1 - \frac{a_p}{p^s} + \left(\frac{\Delta_k}{p}\right) \frac{1}{p^{2s}}.$$

In the case $p \mathcal{O}_{k,f} = \mathfrak{p}_1 \mathfrak{p}_2$, where $\mathfrak{p}_1, \mathfrak{p}_2$ are prime ideals of $\mathcal{O}_{k,f}$, we have $\left(\frac{\Delta_k}{p}\right) = 1$, $\chi(\mathfrak{p}_1) \chi(\mathfrak{p}_2) = 1$ and so $a_p = \chi(\mathfrak{p}_1) + \chi(\mathfrak{p}_2)$. In the case $p \mathcal{O}_{k,f} = \mathfrak{p}$ or \mathfrak{p}^2 , we have $\chi(\mathfrak{p}) = 1$, so

$$\prod_{\mathfrak{p} \mid p} \left(1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s}\right) = \prod_{\mathfrak{p} \mid p} \left(1 - \frac{1}{\mathbf{N}\mathfrak{p}^s}\right) = \left(1 - \frac{1}{p^s}\right) \left(1 - \left(\frac{\Delta_k}{p}\right) \frac{1}{p^{3s}}\right).$$

Thus we may take $a_p = 1 + \left(\frac{\Delta_k}{p}\right)$. The Lemma follows easily from identity (5.5). \square

Remark 5.8. It is interesting to observe that if χ is a non-trivial primitive cubic character of $Cl_{k,f}$, and p is a rational prime not dividing f , then a_p in (5.5) is given by the formula

$$a_p = |\{[u : v] \in \mathbb{P}^1(\mathbb{F}_p) \mid x(u, v) = 0\}| - 1.$$

Here K denotes the isomorphic class of the cubic fields corresponding to χ ,

$$x(u, v) = x_0 u^3 + x_1 u^2 v + x_2 u v^2 + x_3 v^3$$

is an integral primitive binary cubic form belonging to the incomplete canonical class of K (i.e., there exists a generator θ of K such that $x(\theta, 1) = 0$ and $\text{disc}(x) = \Delta_K$, see [8]). In fact, the Hasse-Weil zeta function of the zero dimensional variety defined by $x(u, v) = 0$ is

$$\zeta(s) L(s, \chi, \mathcal{O}_{k,f}) = \zeta_K(s).$$

APPENDIX A. ABELIAN L -SERIES OF NUMBER FIELDS AND L -SERIES OF ORDERS

In this section we let k denote a number field and let $\mathfrak{m} = \mathfrak{m}_0 \mathfrak{m}_\infty$ be a modulus in k . We associate to each character χ of $Cl_k(\mathfrak{m}) = I_k(\mathfrak{m})/P_{k,1}(\mathfrak{m})$ the L -series

$$L_{k,\mathfrak{m}}(s, \chi) = \sum_{(\mathfrak{a}, \mathfrak{m}_0)=1} \chi(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-s},$$

where \mathfrak{a} runs through prime-to- \mathfrak{m}_0 integral ideals of \mathcal{O}_k . We let A denote the étale algebra k^{n+1} for some positive integer n . We write $\mathcal{O}_A = \mathcal{O}_k^{n+1}$ for the maximal order of A and $\mathbf{1}$ for the identity element of A . Observe that the \mathcal{O}_A -ideals have the form $(\mathfrak{a}_0, \dots, \mathfrak{a}_n)$, where the \mathfrak{a}_i 's are fractional ideals of \mathcal{O}_k . We let $\mathcal{O}_{A,\mathfrak{m}}$ denote an order of A of the following form

$$\mathcal{O}_k \mathbf{1} + \mathfrak{m}_0 \mathcal{O}_A = \{(\alpha_0, \dots, \alpha_n) \in \mathcal{O}_A \mid \alpha_i \equiv \alpha_0 \pmod{\mathfrak{m}_0}, 1 \leq i \leq n\}.$$

Note that $\mathcal{O}_{A,\mathfrak{m}}$ is an \mathcal{O}_k -module and has the conductor $\mathfrak{m}_0 \mathcal{O}_A$. Moreover, we say that an element $\alpha = (\alpha_0, \dots, \alpha_n) \in A^*$ is equivariant modulo \mathfrak{m}_∞ if

$$\sigma(\alpha_i / \alpha_0) > 0, \quad \forall 1 \leq i \leq n, \quad \forall \sigma \in \mathfrak{m}_\infty.$$

Put

$$P_{\mathfrak{m}_\infty}(\mathcal{O}_{A,\mathfrak{m}}) = \{\alpha \mathcal{O}_{A,\mathfrak{m}} \mid \alpha \in A^* \text{ is equivariant modulo } \mathfrak{m}_\infty\}.$$

We call the quotient group $I(\mathcal{O}_{A,\mathfrak{m}})/P_{\mathfrak{m}_\infty}(\mathcal{O}_{A,\mathfrak{m}})$ the narrow class group of $\mathcal{O}_{A,\mathfrak{m}}$ modulo \mathfrak{m}_∞ , and denote it by $Cl_{A,\mathfrak{m}}$. To each character $\tilde{\chi}$ of $Cl_{A,\mathfrak{m}}$, we associate the L -series

$$L(s, \tilde{\chi}, \mathcal{O}_{A,\mathfrak{m}}) = \sum_{\tilde{\mathfrak{a}} \subseteq \mathcal{O}_{A,\mathfrak{m}}} \tilde{\chi}(\tilde{\mathfrak{a}}) \mathbf{N}\tilde{\mathfrak{a}}^{-s},$$

where $\tilde{\mathfrak{a}}$ goes through integral invertible ideals of $\mathcal{O}_{A,\mathfrak{m}}$. We also define the truncated L -series

$$L^*(s, \tilde{\chi}, \mathcal{O}_{A,\mathfrak{m}}) = \sum_{(\tilde{\mathfrak{a}}, \mathfrak{m}_0 \mathcal{O}_A)=1} \tilde{\chi}(\tilde{\mathfrak{a}}) \mathbf{N}\tilde{\mathfrak{a}}^{-s},$$

where $\tilde{\mathfrak{a}}$ ranges over integral ideals of $\mathcal{O}_{A,\mathfrak{m}}$ prime to the conductor of $\mathcal{O}_{A,\mathfrak{m}}$.

The goal of this section is to derive a simple relation between the abelian L -series $L_{k,\mathfrak{m}}(s, \chi)$ of k and the L -series $L(s, \tilde{\chi}, \mathcal{O}_{A,\mathfrak{m}})$ of the order $\mathcal{O}_{A,\mathfrak{m}}$.

The relations between $L_{k,m}(s, \chi)$ and the Truncated L -series of $\mathcal{O}_{A,m}$.

Lemma A.1. *Let $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{b}}$ be integral ideals of $\mathcal{O}_{A,m}$ prime to $\mathfrak{m}_0\mathcal{O}_A$. Suppose there exists an $\gamma = (\gamma_0, \dots, \gamma_n) \in A^*$ equivariant modulo \mathfrak{m}_∞ such that $\tilde{\mathfrak{b}} = \gamma \tilde{\mathfrak{a}}$. Then*

$$\gamma_i / \gamma_0 \equiv 1 \pmod{\mathfrak{m}}, \quad \forall 1 \leq i \leq n.$$

Proof. Since $\tilde{\mathfrak{a}}\mathcal{O}_A$ and $\gamma\tilde{\mathfrak{a}}\mathcal{O}_A$ are integral ideals of \mathcal{O}_A prime to $\mathfrak{m}_0\mathcal{O}_A$, each component γ_i of γ is prime to \mathfrak{m}_0 . Moreover, from $\tilde{\mathfrak{a}} + \mathfrak{m}_0\mathcal{O}_A = \mathcal{O}_{A,m}$, we see that there exists a $\alpha = (\alpha_0, \dots, \alpha_n) \in \tilde{\mathfrak{a}}$ such that $\alpha_i\mathcal{O}_k + \mathfrak{m}_0 = \mathcal{O}_k$, $0 \leq i \leq n$. Since α and $\gamma\alpha$ both belong to $\mathcal{O}_{A,m}$, we have $\alpha_i \equiv \alpha_0 \pmod{\mathfrak{m}_0}$ and $\gamma_i \alpha_i \equiv \gamma_0 \alpha_0 \pmod{\mathfrak{m}_0}$, from which we deduce that $\gamma_i / \gamma_0 \equiv 1 \pmod{\mathfrak{m}}$, $\forall 1 \leq i \leq n$. \square

We define a map $\Psi : Cl_{A,m} \rightarrow Cl_k(\mathcal{O}_k) \times Cl_k(\mathfrak{m})^n$ as follows. Let $\tilde{\mathfrak{A}}$ be an ideal class of $Cl_{A,m}$. Choose in $\tilde{\mathfrak{A}}$ an integral $\mathcal{O}_{A,m}$ -ideal $\tilde{\mathfrak{a}}$ prime to $\mathfrak{m}_0\mathcal{O}_A$. Write $\tilde{\mathfrak{a}}\mathcal{O}_A = (\mathfrak{a}_0, \dots, \mathfrak{a}_n)$, where the \mathfrak{a}_i 's are integral ideals of \mathcal{O}_k prime to \mathfrak{m}_0 . Let \mathfrak{A}_0 denote the ideal class of \mathfrak{a}_0 in $Cl_k(\mathcal{O}_k)$, and \mathfrak{A}_i ($1 \leq i \leq n$) the ray class of $\mathfrak{a}_i/\mathfrak{a}_0$ in $Cl_k(\mathfrak{m})$. Then

$$\Psi(\tilde{\mathfrak{A}}) = (\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n) \in Cl_k(\mathcal{O}_k) \times Cl_k(\mathfrak{m})^n.$$

By Lemma A.1, the map Ψ is well defined. It is also clear that Ψ is a surjective homomorphism. Now suppose $\Psi(\tilde{\mathfrak{A}}) = 1$, that is, $\tilde{\mathfrak{A}}$ is represented by some integral $\mathcal{O}_{A,m}$ -ideal $\tilde{\mathfrak{a}}$ with $\tilde{\mathfrak{a}}\mathcal{O}_A = (\gamma_0\mathcal{O}_k, \dots, \gamma_n\mathcal{O}_k)$, where the γ_i 's are elements of \mathcal{O}_k prime to \mathfrak{m}_0 such that $\gamma_i/\gamma_0 \equiv 1 \pmod{\mathfrak{m}}$, $0 \leq i \leq n$. Then $\gamma = (\gamma_0, \dots, \gamma_n)$ is in $\mathcal{O}_{A,m}$ and equivariant modulo \mathfrak{m}_∞ . Since $\tilde{\mathfrak{a}}$ and $\gamma\mathcal{O}_{A,m}$ are both integral ideals of $\mathcal{O}_{A,m}$ prime to $\mathfrak{m}_0\mathcal{O}_A$ and under $\tilde{\mathfrak{a}}\mathcal{O}_A$, Lemma 2.1 implies that $\tilde{\mathfrak{a}} = \gamma\mathcal{O}_{A,m}$. This shows that Ψ is injective.

Lemma A.2. *The mapping $\Psi : Cl_{A,m} \rightarrow Cl_k(\mathcal{O}_k) \times Cl_k(\mathfrak{m})^n$ is an isomorphism.*

Next let $\tilde{\mathfrak{A}} \in Cl_{A,m}$ and suppose $\Psi(\tilde{\mathfrak{A}}) = (\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, where $\mathfrak{A}_0 \in Cl_k(\mathcal{O}_k)$ and $\mathfrak{A}_i \in Cl_k(\mathfrak{m})$, $1 \leq i \leq n$. As before, we associate to $\tilde{\mathfrak{A}}$ the truncated partial zeta function

$$\zeta^*(s, \tilde{\mathfrak{A}}, \mathcal{O}_{A,m}) = \sum_{\substack{\tilde{\mathfrak{a}} \in \tilde{\mathfrak{A}} \\ (\tilde{\mathfrak{a}}, \mathfrak{m}_0\mathcal{O}_A)=1}} N\tilde{\mathfrak{a}}^{-s},$$

where $\tilde{\mathfrak{a}}$ ranges over those integral $\mathcal{O}_{A,m}$ -ideals prime to $\mathfrak{m}_0\mathcal{O}_A$ lying in the class $\tilde{\mathfrak{A}}$. Moreover, for each ray class $\mathfrak{A} \in Cl_k(\mathfrak{m})$, put

$$\zeta_{k,m}(s, \mathfrak{A}) = \sum_{\substack{\mathfrak{a} \in \mathfrak{A} \\ (\mathfrak{a}, \mathfrak{m}_0)=1}} N\mathfrak{a}^{-s}.$$

Lemma A.3. *Let $\tilde{\mathfrak{A}}$ and the \mathfrak{A}_i 's be as above. Suppose that the class $\mathfrak{A}_0 \in Cl_k(\mathcal{O}_k)$ breaks up into finitely many smaller ray classes modulo \mathfrak{m} , say, $\mathfrak{B}_1, \dots, \mathfrak{B}_h$, where $h = |Cl_k(\mathfrak{m})| / |Cl_k(\mathcal{O}_k)|$. Then we have*

$$\zeta^*(s, \tilde{\mathfrak{A}}, \mathcal{O}_{A,m}) = \sum_{j=1}^h \zeta_{k,m}(s, \mathfrak{B}_j) \zeta_{k,m}(s, \mathfrak{B}_j \mathfrak{A}_1) \cdots \zeta_{k,m}(s, \mathfrak{B}_j \mathfrak{A}_n).$$

Proof. Let $\tilde{\mathfrak{a}} \in \tilde{\mathfrak{A}}$ be an integral $\mathcal{O}_{A,\mathfrak{m}}$ -ideal prime to $\mathfrak{m}_0 \mathcal{O}_A$. Put $\tilde{\mathfrak{a}} \mathcal{O}_A = (\mathfrak{a}_0, \dots, \mathfrak{a}_n)$, where the \mathfrak{a}_i 's are integral ideals of \mathcal{O}_k prime to \mathfrak{m}_0 . Then $\mathbf{N}\tilde{\mathfrak{a}} = \mathbf{N}\mathfrak{a}_0 \cdots \mathbf{N}\mathfrak{a}_n$. Moreover, we have $\mathfrak{a}_0 \in \mathfrak{B}_j$ for a unique j , $1 \leq j \leq h$, and $\mathfrak{a}_i \in \mathfrak{B}_j \mathfrak{A}_i$ for $1 \leq i \leq n$. Conversely, given $1 \leq j \leq h$, and given integral \mathcal{O}_k -ideals $\mathfrak{a}_0, \dots, \mathfrak{a}_n$ prime to \mathfrak{m}_0 such that $\mathfrak{a}_0 \in \mathfrak{B}_j$, $\mathfrak{a}_i \in \mathfrak{B}_j \mathfrak{A}_i$ for $1 \leq i \leq n$, there exists a unique integral $\mathcal{O}_{A,\mathfrak{m}}$ -ideal $\tilde{\mathfrak{a}} \in \tilde{\mathfrak{A}}$ prime to $\mathfrak{m}_0 \mathcal{O}_A$ under the \mathcal{O}_A -ideal $(\mathfrak{a}_0, \dots, \mathfrak{a}_n)$. Hence we have

$$\zeta^*(s, \tilde{\mathfrak{A}}, \mathcal{O}_{A,\mathfrak{m}}) = \sum_{j=1}^h \sum_{\substack{\mathfrak{a}_0 \in \mathfrak{B}_j \\ (\mathfrak{a}_0, \mathfrak{m}_0)=1}} \mathbf{N}\mathfrak{a}_0^{-s} \sum_{\substack{\mathfrak{a}_1 \in \mathfrak{B}_j \mathfrak{A}_1 \\ (\mathfrak{a}_1, \mathfrak{m}_0)=1}} \mathbf{N}\mathfrak{a}_1^{-s} \cdots \sum_{\substack{\mathfrak{a}_n \in \mathfrak{B}_j \mathfrak{A}_n \\ (\mathfrak{a}_n, \mathfrak{m}_0)=1}} \mathbf{N}\mathfrak{a}_n^{-s}.$$

□

Let $\tilde{\chi}$ be a character of $Cl_{A,\mathfrak{m}}$. Through the isomorphism Ψ , we may identify $\tilde{\chi}$ as a character of $Cl_k(\mathcal{O}_k) \times Cl_k(\mathfrak{m})^n$. Thus $\tilde{\chi} = \chi_0 \times \chi_1 \times \cdots \times \chi_n$, where χ_0 is a character of $Cl_k(\mathcal{O}_k)$ and χ_i ($1 \leq i \leq n$) are characters of $Cl_k(\mathfrak{m})$.

Theorem A.4. *With notation as above, we have*

$$(A.1) \quad L^*(s, \tilde{\chi}, \mathcal{O}_{A,\mathfrak{m}}) = L_{k,\mathfrak{m}}(s, \chi_1) \cdots L_{k,\mathfrak{m}}(s, \chi_n) L_{k,\mathfrak{m}}(s, \chi_0 \chi_1^{-1} \cdots \chi_n^{-1}),$$

where in the last L -function we identify χ_0 with its induced character at $Cl_k(\mathfrak{m})$ through the surjective homomorphism $Cl_k(\mathfrak{m}) \rightarrow Cl_k(\mathcal{O}_k)$.

Proof. Lemma A.3 implies that

$$\begin{aligned} L^*(s, \tilde{\chi}, \mathcal{O}_{A,\mathfrak{m}}) &= \sum_{\mathfrak{A}_0 \in Cl_k(\mathcal{O}_k)} \sum_{\mathfrak{A}_1 \in Cl_k(\mathfrak{m})} \cdots \sum_{\mathfrak{A}_n \in Cl_k(\mathfrak{m})} \chi_0(\mathfrak{A}_0) \chi_1(\mathfrak{A}_1) \cdots \chi_n(\mathfrak{A}_n) \\ &\quad \sum_{\substack{\mathfrak{B} \in Cl_k(\mathfrak{m}) \\ \mathfrak{B} \subseteq \mathfrak{A}_0}} \zeta_{k,\mathfrak{m}}(s, \mathfrak{B}) \zeta_{k,\mathfrak{m}}(s, \mathfrak{B} \mathfrak{A}_1) \cdots \zeta_{k,\mathfrak{m}}(s, \mathfrak{B} \mathfrak{A}_n). \end{aligned}$$

Regarding χ_0 as a character defined at $Cl_k(\mathfrak{m})$, we may write the above as

$$\begin{aligned} \sum_{\mathfrak{B} \in Cl_k(\mathfrak{m})} \chi_0 \chi_1^{-1} \cdots \chi_n^{-1}(\mathfrak{B}) \zeta_{k,\mathfrak{m}}(s, \mathfrak{B}) &\sum_{\mathfrak{A}_1 \in Cl_k(\mathfrak{m})} \chi_1(\mathfrak{B} \mathfrak{A}_1) \zeta_{k,\mathfrak{m}}(s, \mathfrak{B} \mathfrak{A}_1) \\ &\cdots \sum_{\mathfrak{A}_n \in Cl_k(\mathfrak{m})} \chi_n(\mathfrak{B} \mathfrak{A}_n) \zeta_{k,\mathfrak{m}}(s, \mathfrak{B} \mathfrak{A}_n). \end{aligned}$$

This is the right hand side of (A.1). □

Corollary A.5. *Let χ be a character of $Cl_k(\mathfrak{m})$ and let $\tilde{\chi}$ be the character of $Cl_{A,\mathfrak{m}}$ corresponding to $1 \times \chi \times \cdots \times \chi$ under Ψ . Then we have*

$$L_{k,\mathfrak{m}}(s, \chi)^n L_{k,\mathfrak{m}}(s, \chi^{-n}) = L^*(s, \tilde{\chi}, \mathcal{O}_{A,\mathfrak{m}}).$$

In particular, if we choose n to be the order of χ , then we get

$$L_{k,\mathfrak{m}}(s, \chi)^n L_{k,\mathfrak{m}}(s, 1) = L^*(s, \tilde{\chi}, \mathcal{O}_{A,\mathfrak{m}}).$$

Theorem A.4 allows us to compute $L^*(s, \tilde{\chi}, \mathcal{O}_{A,\mathfrak{m}})$ from the $L_{k,\mathfrak{m}}(s, \chi)$'s and vice versa. In particular, let $A = \mathbb{Q}^2$, $m \in \mathbb{Z}^+$ and $\mathcal{O}_{A,m} = \mathbb{Z} \mathbf{1} + m \mathcal{O}_A$. Then each character $\tilde{\chi}$ of $Cl_{A,m}$ corresponds to $1 \times \chi$ for a unique character χ of $Cl_{\mathbb{Q}}(m)$ so that $\tilde{\chi}$ is primitive if and only if χ is primitive. Moreover, we have

$$L^*(s, \tilde{\chi}, \mathcal{O}_{A,m}) = L_{\mathbb{Q},m}(s, \chi) L_{\mathbb{Q},m}(s, \chi^{-1}).$$

This result is used in Section 4.

The relations between the L -series and the Truncated L -series of $\mathcal{O}_{A,m}$. For the rest of the section we let $\mathfrak{m}' = \mathfrak{m}'_0 \mathfrak{m}_\infty$ denote a modulus in k whose finite part \mathfrak{m}'_0 is an integral ideal of \mathcal{O}_k dividing \mathfrak{m}_0 . We define $\mathcal{O}_{A,m'}$ and its narrow class group $Cl_{A,m'}$ modulo \mathfrak{m}_∞ as before. For such a modulus \mathfrak{m}' , we let $\varphi_{A,m',m} : Cl_{A,m} \rightarrow Cl_{A,m'}$ denote the homomorphism sending the narrow ideal class of $\tilde{\mathfrak{a}}$ modulo \mathfrak{m}_∞ to the narrow ideal class of $\tilde{\mathfrak{a}} \mathcal{O}_{A,m'}$ modulo \mathfrak{m}_∞ , for every $\tilde{\mathfrak{a}} \in I(\mathcal{O}_{A,m})$. Let $U_{A,m'}$ denote the group formed by units of $\mathcal{O}_{A,m'}$ which are equivariant modulo \mathfrak{m}_∞ . It is easy to see that $\varphi_{A,m',m}$ is surjective and induces the exact sequence

$$1 \longrightarrow U_{A,m} \longrightarrow U_{A,m'} \longrightarrow U(\mathcal{O}_{A,m'}, \mathcal{O}_{A,m}) \longrightarrow Cl_{A,m} \longrightarrow Cl_{A,m'} \longrightarrow 1.$$

Thus we have

$$(A.2) \quad |U(\mathcal{O}_{A,m'}, \mathcal{O}_{A,m})| = [U_{A,m'} : U_{A,m}] |Cl_{A,m}| / |Cl_{A,m'}|.$$

Lemma A.6. *For each class $\tilde{\mathfrak{A}}$ of $Cl_{A,m}$, we have*

$$\zeta(s, \tilde{\mathfrak{A}}, \mathcal{O}_{A,m}) = \sum_{\substack{\mathfrak{c} \mathfrak{m}'_0 = \mathfrak{m}_0}} \frac{[U_{A,m'} : U_{A,m}]}{\mathbf{N}\mathfrak{c}^{(n+1)s}} \zeta^*(s, [\mathfrak{c}^{-1} \mathcal{O}_{A,m'}] \varphi_{A,m',m}(\tilde{\mathfrak{A}}), \mathcal{O}_{A,m'}),$$

where \mathfrak{m}' denotes the modulus $\mathfrak{m}'_0 \mathfrak{m}_\infty$ and the sum runs over the pairs of integral \mathcal{O}_k -ideals \mathfrak{c} and \mathfrak{m}'_0 with $\mathfrak{c} \mathfrak{m}'_0 = \mathfrak{m}_0$.

Proof. Let $\tilde{\mathfrak{a}}$ be an integral invertible $\mathcal{O}_{A,m}$ -ideal in the class $\tilde{\mathfrak{A}}$. Let \mathfrak{c} and \mathfrak{m}'_0 be the integral \mathcal{O}_k -ideals given by $\mathfrak{c} \mathbf{1} = (\tilde{\mathfrak{a}} + \mathfrak{m}_0 \mathcal{O}_A) \cap \mathcal{O}_k \mathbf{1}$ and $\mathfrak{m}'_0 = \mathfrak{m}_0 / \mathfrak{c}$. Put $\mathfrak{m}' = \mathfrak{m}_\infty \mathfrak{m}'_0$. Then by modular law (see Lemma 5.2), we have $\tilde{\mathfrak{a}} + \mathfrak{m}_0 \mathcal{O}_A = \mathfrak{c} \mathcal{O}_{A,m'}$. Thus $\tilde{\mathfrak{b}} = \mathfrak{c}^{-1} \tilde{\mathfrak{a}} \mathcal{O}_{A,m'}$ is an integral $\mathcal{O}_{A,m'}$ -ideal prime to $\mathfrak{m}'_0 \mathcal{O}_A$ lying in the class $[\mathfrak{c}^{-1} \mathcal{O}_{A,m'}] \varphi_{A,m',m}(\tilde{\mathfrak{A}})$. Moreover, we have $\mathbf{N}\tilde{\mathfrak{a}} = \mathbf{N}\mathfrak{c}^{n+1} \mathbf{N}\tilde{\mathfrak{b}}$. Conversely, given integral \mathcal{O}_k -ideals \mathfrak{c} and \mathfrak{m}'_0 with $\mathfrak{c} \mathfrak{m}'_0 = \mathfrak{m}_0$, and given any integral $\mathcal{O}_{A,m'}$ -ideal $\tilde{\mathfrak{b}}$ prime to the conductor of $\mathcal{O}_{A,m'}$ (here $\mathfrak{m}' = \mathfrak{m}'_0 \mathfrak{m}_\infty$), all the invertible $\mathcal{O}_{A,m}$ -ideals $\tilde{\mathfrak{a}}$ under $\mathfrak{c} \tilde{\mathfrak{b}}$ are integral ideals of $\mathcal{O}_{A,m}$ and satisfy

$$(\tilde{\mathfrak{a}} + \mathfrak{m}_0 \mathcal{O}_A) \cap \mathcal{O}_k \mathbf{1} = \mathfrak{c} \mathbf{1}$$

(see the proof of Lemma 5.3). Suppose further that $\tilde{\mathfrak{b}}$ lies in $[\mathfrak{c}^{-1} \mathcal{O}_{A,m'}] \varphi_{A,m',m}(\tilde{\mathfrak{A}})$. This means that there exists some $\mathcal{O}_{A,m}$ -invertible $\tilde{\mathfrak{a}} \in \tilde{\mathfrak{A}}$ such that $\tilde{\mathfrak{a}} \mathcal{O}_{A,m'} = \mathfrak{c} \tilde{\mathfrak{b}}$. Then under $\mathfrak{c} \tilde{\mathfrak{b}}$ there are precisely $[U_{A,m'} : U_{A,m}]$ invertible $\mathcal{O}_{A,m}$ -ideals lying in the class $\tilde{\mathfrak{A}}$. They are given by $\varepsilon \tilde{\mathfrak{a}}$, where ε goes over a complete set of representatives of $U_{A,m'}$ modulo $U_{A,m}$. The Lemma follows by grouping the terms in $\zeta(s, \tilde{\mathfrak{A}}, \mathcal{O}_{A,m})$ according to $(\tilde{\mathfrak{a}} + \mathfrak{m}_0 \mathcal{O}_A) \cap \mathcal{O}_k \mathbf{1}$. \square

In the following we let $\tilde{\chi}$ be a character of $Cl_{A,m}$. Through the isomorphism Ψ , we identify $\tilde{\chi}$ as a character of $Cl_k(\mathcal{O}_k) \times Cl_k(\mathfrak{m})^n$ and write $\tilde{\chi} = \chi_0 \times \chi_1 \times \cdots \times \chi_n$, where χ_0 is a character of $Cl_k(\mathcal{O}_k)$ and χ_i ($1 \leq i \leq n$) are characters of $Cl_k(\mathfrak{m})$. Let \mathfrak{f}_i denote the finite part of the conductor of χ_i , $1 \leq i \leq n$. Let \mathfrak{f} denote the least common multiple ideal of $\mathfrak{f}_1, \dots, \mathfrak{f}_n$. It is easy to see that $\tilde{\chi}$ is defined at $Cl_{A,m'}$ with $\mathfrak{m}' = \mathfrak{m}'_0 \mathfrak{m}_\infty$ if and only if $\mathfrak{f} | \mathfrak{m}'_0$ and $\mathfrak{m}'_0 | \mathfrak{m}_0$. As before, we use the same symbol $\tilde{\chi}$ to denote the character of $Cl_{A,m'}$ which induces $\tilde{\chi}$.

Theorem A.7. *With notation as above, we have*

$$L(s, \tilde{\chi}, \mathcal{O}_{A, \mathfrak{m}}) = \sum_{\mathfrak{f} | \mathfrak{m}'_0 | \mathfrak{m}_0} \chi_0(\mathfrak{c}) \frac{|U(\mathcal{O}_{A, \mathfrak{m}'}, \mathcal{O}_{A, \mathfrak{m}})|}{\mathbf{N}\mathfrak{c}^{(n+1)s}} L^*(s, \tilde{\chi}, \mathcal{O}_{A, \mathfrak{m}'}) ,$$

where $\mathfrak{m}' = \mathfrak{m}'_0 \mathfrak{m}_\infty$, $\mathfrak{c} = \mathfrak{m}_0 / \mathfrak{m}'_0$ and the sum extends over integral \mathcal{O}_k -ideals \mathfrak{m}'_0 such that $\mathfrak{f} | \mathfrak{m}'_0$ and $\mathfrak{m}'_0 | \mathfrak{m}_0$. In particular, if $\mathfrak{f} = \mathfrak{m}_0$, then we have

$$L(s, \tilde{\chi}, \mathcal{O}_{A, \mathfrak{m}}) = L^*(s, \tilde{\chi}, \mathcal{O}_{A, \mathfrak{m}}) .$$

Proof. By Lemma A.6, we can write $L(s, \tilde{\chi}, \mathcal{O}_{A, \mathfrak{m}})$ as

$$\begin{aligned} & \sum_{\tilde{\mathfrak{A}} \in Cl_{A, \mathfrak{m}}} \tilde{\chi}(\tilde{\mathfrak{A}}) \sum_{\mathfrak{c} \mathfrak{m}'_0 = \mathfrak{m}_0} \frac{[U_{A, \mathfrak{m}'} : U_{A, \mathfrak{m}}]}{\mathbf{N}\mathfrak{c}^{(n+1)s}} \zeta^*(s, [\mathfrak{c}^{-1} \mathcal{O}_{A, \mathfrak{m}'}] \varphi_{A, \mathfrak{m}', \mathfrak{m}}(\tilde{\mathfrak{A}}), \mathcal{O}_{A, \mathfrak{m}'}) \\ &= \sum_{\mathfrak{c} \mathfrak{m}'_0 = \mathfrak{m}_0} \frac{[U_{A, \mathfrak{m}'} : U_{A, \mathfrak{m}}]}{\mathbf{N}\mathfrak{c}^{(n+1)s}} \sum_{\tilde{\mathfrak{A}}' \in Cl_{A, \mathfrak{m}'}} \sum_{\substack{\tilde{\mathfrak{A}} \in Cl_{A, \mathfrak{m}} \\ \varphi_{A, \mathfrak{m}', \mathfrak{m}}(\tilde{\mathfrak{A}}) = [\mathfrak{c} \mathcal{O}_{A, \mathfrak{m}'}] \tilde{\mathfrak{A}}'}} \tilde{\chi}(\tilde{\mathfrak{A}}) \zeta^*(s, \tilde{\mathfrak{A}}', \mathcal{O}_{A, \mathfrak{m}'}). \end{aligned}$$

For each $\tilde{\mathfrak{A}}' \in Cl_{A, \mathfrak{m}'}$, there is a $\tilde{\mathfrak{A}}^* \in Cl_{A, \mathfrak{m}}$ such that $\varphi_{A, \mathfrak{m}', \mathfrak{m}}(\tilde{\mathfrak{A}}^*) = [\mathfrak{c} \mathcal{O}_{A, \mathfrak{m}'}] \tilde{\mathfrak{A}}'$. So we have

$$\begin{aligned} \sum_{\substack{\tilde{\mathfrak{A}} \in Cl_{A, \mathfrak{m}} \\ \varphi_{A, \mathfrak{m}', \mathfrak{m}}(\tilde{\mathfrak{A}}) = [\mathfrak{c} \mathcal{O}_{A, \mathfrak{m}'}] \tilde{\mathfrak{A}}'}} \tilde{\chi}(\tilde{\mathfrak{A}}) &= \tilde{\chi}(\tilde{\mathfrak{A}}^*) \sum_{\tilde{\mathfrak{B}} \in \text{Ker } \varphi_{A, \mathfrak{m}', \mathfrak{m}}} \tilde{\chi}(\tilde{\mathfrak{B}}) \\ &= \begin{cases} \tilde{\chi}(\tilde{\mathfrak{A}}^*) |\text{Ker } \varphi_{A, \mathfrak{m}', \mathfrak{m}}|, & \text{if } \tilde{\chi} \text{ is trivial on } \text{Ker } \varphi_{A, \mathfrak{m}', \mathfrak{m}}; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The sum above vanishes unless $\tilde{\chi}$ is defined at $Cl_{A, \mathfrak{m}'}$. Thus, using (A.2), we get

$$L(s, \tilde{\chi}, \mathcal{O}_{A, \mathfrak{m}}) = \sum_{\substack{\mathfrak{c} \mathfrak{m}'_0 = \mathfrak{m}_0 \\ \mathfrak{f} | \mathfrak{m}'_0}} \tilde{\chi}(\mathfrak{c} \mathcal{O}_{A, \mathfrak{m}'}) \frac{|U(\mathcal{O}_{A, \mathfrak{m}'}, \mathcal{O}_{A, \mathfrak{m}})|}{\mathbf{N}\mathfrak{c}^{(n+1)s}} \sum_{\tilde{\mathfrak{A}}' \in Cl_{A, \mathfrak{m}'}} \tilde{\chi}(\tilde{\mathfrak{A}}') \zeta^*(s, \tilde{\mathfrak{A}}', \mathcal{O}_{A, \mathfrak{m}'}),$$

the Theorem follows. \square

Remark A.8. It is easy to prove that

$$|U(\mathcal{O}_{A, \mathfrak{m}'}, \mathcal{O}_{A, \mathfrak{m}})| = \left(\mathbf{N}(\mathfrak{m}_0 / \mathfrak{m}'_0) \prod_{\substack{\mathfrak{p} | \mathfrak{m}_0 \\ \mathfrak{p} \nmid \mathfrak{m}'_0}} \left(1 - \frac{1}{\mathbf{N}\mathfrak{p}}\right) \right)^n .$$

Let μ denote the Möbius function on integral ideals of \mathcal{O}_k , that is, μ is a function satisfying $\mu(\mathcal{O}_k) = 1$, $\mu(\mathfrak{p}) = -1$ for each prime ideal \mathfrak{p} of \mathcal{O}_k , and $\mu(\mathfrak{p}^l) = 0$ for all integers $l > 1$. Moreover, $\mu(\mathfrak{a})\mu(\mathfrak{b}) = \mu(\mathfrak{a}\mathfrak{b})$ if \mathfrak{a} and \mathfrak{b} are coprime integral ideals. The Möbius function μ satisfies the relation

$$\sum_{\mathfrak{c} | \mathfrak{c}'} \mu(\mathfrak{c}) = \begin{cases} 1, & \text{if } \mathfrak{c}' = \mathcal{O}_k; \\ 0, & \text{otherwise.} \end{cases}$$

for all integral ideals \mathfrak{c}' of \mathcal{O}_k . From this relation we deduce immediately the following inversion formula.

Corollary A.9. *With notation as above, we have*

$$L^*(s, \tilde{\chi}, \mathcal{O}_{A, \mathfrak{m}}) = \sum_{\mathfrak{f} | \mathfrak{m}'_0 | \mathfrak{m}_0} \mu(\mathfrak{c}) \chi_0(\mathfrak{c}) \frac{|U(\mathcal{O}_{A, \mathfrak{m}'}, \mathcal{O}_{A, \mathfrak{m}})|}{\mathbf{N} \mathfrak{c}^{(n+1)s}} L(s, \tilde{\chi}, \mathcal{O}_{A, \mathfrak{m}'}),$$

where in the sum above $\mathfrak{m}' = \mathfrak{m}'_0 \mathfrak{m}_\infty$ and $\mathfrak{c} = \mathfrak{m}_0 / \mathfrak{m}'_0$.

ACKNOWLEDGMENT

The author would like to thank Frank Thorne for helpful comments related to the topics of this paper.

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